Decomposavle Semi-Regenerative Processes and their Applications

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ABSTRACT

The talk consists from two parts. In the first one a short review of the Smith's regenerative idea development is proposed. Decomposable Semi-Regenerative Processes (DSRPr) as its generalization is proposed. Some previous its applications is reviewed.

In the second part of the talk the DSRPr is used for study of reliability of a double redundant reparable system with arbitrary distributed life and repair times of its components

Outline

A little history I

- Apparently Ja. Bernully was the first, who consider i.i.d. r.v.'s (1713) [1]
- W. Smith in (1955) [2] spread out this notion to processes in time and propose the nation of a **Regenerative process**.

We start with the very known notion of **regeneration** [Smith (1955)]. Let $X = \{X(t),\ t \in \mathrm{R}\}$ be a Stochastic Process (SP) with measurable states space $(E, \mathcal{E})$, and $\mathcal{F}_t^X = \sigma\{X(v),\ v \leq t\}$ be appropriate σ-algebra.

Regenerative Processes. Definition I

A Random variable (r.v.) $S \in \mathcal{F}_t^X$ is called a **Markov Time** (MT), if the event $\{S \le t\}$ became known from observation of the process X trajectory before time t, $\{S \le t\} \in \mathcal{F}_t^X$. A MT S of the SP X is its **Regeneration Time** (RT), if the process "forget" its past at this time,

$$\mathsf{P}\{X(S+t) \in \Gamma \mid \mathcal{F}_S\} = \mathsf{P}\{X(t) \in \Gamma\} \quad \text{for every} \quad \Gamma \in \mathcal{E}. \tag{1}$$

Note that if there exists with probability one a finite RT S, so there exists a sequence of such times $\{S_n,\ n = 1, 2, \dots\}$.

Regenerative Processes. Definition II. Remarks

SP X is called (homogeneous) **Regenerative Process** (RPr), if it "forget" its past at any RT, and its behavior after it is stochastic the same after any RT S_n,

$$\begin{aligned} \mathsf{P}\{X(S_n+t)\in\Gamma\,|\mathcal{F}^X_{S_n}\} &= \mathsf{P}\{X(S_n+t)\in\Gamma\,|S_nS_n \text{ is RT}\} = \\ &= \mathsf{P}\{X(S_1+t)\in\Gamma\} \end{aligned} \tag{2}$$

Intervals $[S_n,\, S_{n+1})$ and their length $T_n = S_{n+1} - S_n$, $n = 0, 1, 2, \dots$ are called **Regeneration Periods** (RP).

Remark 1. Note, that the functional elements $W_n = \{X(S_n + t),\ t \le T_n\}$ are independent. They are called **Regeneration Cycles** (RC).

Remark 2. Sometimes it is necessary to consider a RPr with delay, for which the distribution of the first RC differs from others.

Structure RPr

At the figure 1 the structure of RP of some RPr is shown.

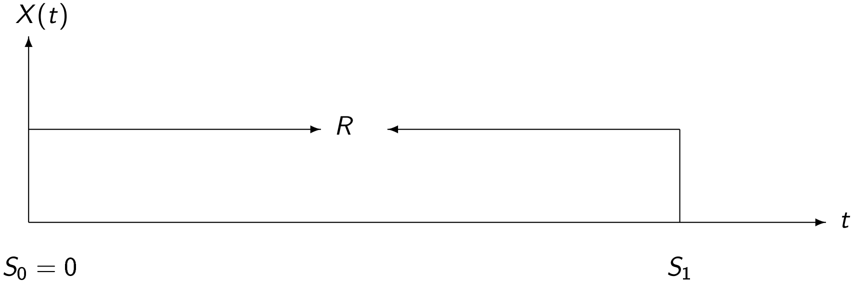

Figure 1: Structure of RP

Notations

Denote by

$$\pi^{(1)}(t,\Gamma) = \mathrm{P}\{X(S_n + t) \in \Gamma,\ t < T_n\}$$

distribution of a RPr at separate RP and by $F(t) = \mathrm{P}\{T_n \le t\}$ and

$$H(t) = \mathrm{E}\left[\sum_{n\ge 0} 1_{\{S_n \le t\}}\right] = \sum_{n\ge 0} \mathrm{P}\{S_n \le t\} = \sum_{n\ge 0} F^{*n}(t)$$

appropriate cumulative distribution function (c.d.f.), and the **Renewal Function** (RF) of the process.

RPr probability distribution and Key Renewal Theorem

With the help of the Complete Probability Formula the RPr distribution can be represented in terms of its distributions at separate RP and the RF in the form

$$\begin{aligned}\pi(t;\,\Gamma) &\equiv \mathsf{P}\{X(t)\in\Gamma\} = \int_0^t \mathsf{P}\{X(t-u)\in\Gamma,\ t-u<T\}H(du)\equiv \\ &\equiv \int_0^t \pi^{(1)}(t-u,\,\Gamma)H(du).\end{aligned}$$

The Key Renewal Theorem [Smith (1955)] allows to prove the existence of stationary probability distribution of the RPr and represent it in the form

$$\pi(\Gamma) = \lim_{t\to\infty}\pi(t;\,\Gamma) = \frac{1}{\mathsf{M}[T_n]}\int_0^\infty \pi^{(1)}(t,\Gamma)dt. \qquad (3)$$

Further History

- A. Markov in (1906) [3] introduce the notion of **Markov dependence** and consider Markov chains
- Apparently E. Cinlar in (1969) [4] and J. Jacod (1971) [5] introduced and investigated **semi-Markov chains**
- Further it leads to construction the theory of **semi-Markov processes** (see [6], and other)

Joining of this notions with Regenerative one leads to introduction of **semi-Regenerative Processes** that firstly appears under different titles

- **Semi-Markov processes with additional trajectories** G. Klimov (1966) [7].
- **Regenerative processes with several types of regeneration points** V. Rykov, M.Yastreenetsky (1971) [8]

Further due to E. Numellin (1978) [9] this notion get the term **Semi-Regenerative Processes**

Semi-Regenerative processes

In applications RT's are usually some state destination times. This state is called **Regeneration State** (RS). In the case of several RS's the notion of RPr is generalized to Semi-egenerative Process (SRPr). Denote by E_1 the set of RS's of the process. In the following it is supposed that it is discrete (finite or denumerable).

Let

- $X = \{X(t),\ t \in \mathrm{R}\}$ be a SP with measurable state space $(E,\ \mathcal{E})$,
- $\mathcal{F}_t^X = \sigma\{X(v),\ \ v \leq t\}$ be generated flow of σ-algebras, and
- $\{S_n,\ n = 0, 1, 2, \ldots\}$ be a sequence of its MT's with $S_0 = 0$.

Definition

(Rykov, Yastrebenetsky, 1971 [8]). A pair $\{X(t), S_n\}$ is called a (homogeneous) **Semi-Regenerative Process** (SRPr), if for any subset $\Gamma \subset \mathcal{E}$ and for all $n = 1, 2, \ldots$ it takes place

$$\begin{aligned} \mathsf{P}\{X(S_n + t) \in \Gamma \,|\mathcal{F}_{S_n}\} &= \mathsf{P}\{X(S_n + t) \in \Gamma|\, X(S_n)\} = \\ &= \mathsf{P}\{X(S_1 + t) \in \Gamma|\, X(S_1)\}. \end{aligned} \quad (4)$$

At that

- r.v.'s S_n are called **Regeneration Times** (RT's),
- intervals $T_n = (S_n,\ S_{n+1}]$ and their lengths $T_n = S_{n+1} - S_n$ are called **Regeneration Periods** (RP's),
- functional random elements $W_n = \{(X(S_n + t), /, T_n),\ t \leq T_n,\ n = 1, 2, \ldots\}$ are called **Regeneration Cycles** (RC), and
- random elements $X_n = X(S_n)$ are called **Regeneration States** (RS).

Regular SRPr

Due to monotonicity of the sequence $\{S_n\}$ there exists a limit $\lim_{n\to\infty} S_n$.

SRPr $\{X(t),\ S_n\}$ is called **regular**, if it exists over all time axis,

$$\mathrm{P}\{\lim_{n\to\infty} S_n = \infty\} = 1.$$

Remark

RC's of SRPr

$$W_n = \{(X(S_{n-1} + t), T_n),\ t \le T_n,\ n = 1, 2, \dots\}$$

form a Markov Chain in functional space, and a regular SPPr is reconstructed (up to equivalency) by it.

Remark

Analogously to usual (in sense of Smith) RPr's sometimes it is necessary to consider also SRPr with delay, for which the distribution of the first RC differs from others. This situation will specially mark if necessary.

Structure SRPr

The structure of a RP's of some RPr at the figure 2 is shown.

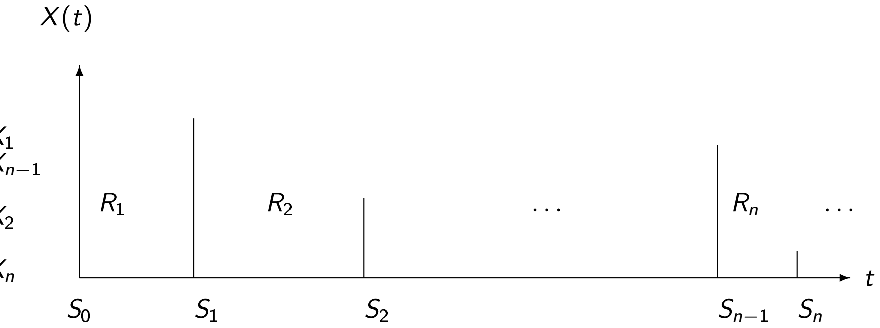

Figure 2: Structure of SRPr

Auxiliary Processes I

In order to investigate SRPr distribution and its characteristics consider the following sequence and process

$$X_n = X(S_n), \quad Y_n = (X_n, T_n) \quad \text{and} \quad N(t) = \max\{n : \ S_n \le t\}.$$

Theorem

Let $\{X(t),\ S_n\}$ is an SRPr. Then the sequences $\{X_n\}$ and $\{Y_n\}$ are homogeneous Markov (MCh) and Semi-Markov chains (SMCh), and $\{N(t)\}$ is a Markov Renewal Process (MRP).

Proof.

See for example in [Jolkof, Rykov (1981) [11]] and [Rykov (1997) [?]]. □

Auxiliary Processes II

Denote by $P = P(x, y)$ and $Q = Q(x, t, y)$ the Transition Matrix (TM) of MCh $\{X_n\}$ and Semi-Markov Matrix (SMM) of SMCh $\{Y_n\}$,

$$\begin{aligned} P(x, y) &= \mathrm{P}\{X_{n+1} = y | X_n = x\}, \\ Q(x, t, y) &= \mathrm{P}\{X_{n+1} = y, T_{n+1} \leq t | X_n = x\}. \end{aligned}$$

Behavior of a SRPr $\{X(t), S_n\}$ mostly determined by the properties of these chains.

Especially, because the very important property of regularity of SRPr $\{X(t),\ S_n\}$ is formulated in an abstract form one needs in the check-able conditions of this property. It could be done in terms of its SMM Q using the following condition of the SRPr *uniform regularity*.

Assumption 1. There are real numbers $a > 0,\ b > 0$ such that for all $x \in E$ the following inequality takes place

$$\sum_{y \in E} Q(x, a, y) < 1 - b < 1. \tag{5}$$

In words this assumption means that the appropriate SMP stays in any state positive time with positive probability.

Theorem

The condition (5) of a SRPr $\{X(t),\ S_n\}$ uniform regularity is sufficient for its regularity.

For proof see Jolkof, Rykov, 1981 [11] and Rykov, 1997 [?].

Distribution of SRP

Denote by

$$H(x,t,y) = \mathsf{E}_x \sum_{n\geq 0} 1_{\{([0,t],\,y)\}}(S_n, X_n)$$

Markov Renewal Matrix (MRM). For the SRPr distribution the following representation takes place

$$\pi(x,t,\Gamma) = \int_0^t \sum_{y\in E} H(x,du,y)\pi^{(1)}(y,t-u,\Gamma) \equiv H \star \pi^{(1)}(x,t,\Gamma), \qquad (6)$$

where a symbol $\star$ denotes the matrix-functional convolution, and

$$\begin{aligned} \pi(x,t,\Gamma) &= \mathsf{P}\{X(t)\in\Gamma \,|\, X(0)=x\}, \\ \pi^{(1)}(x,t,\Gamma) &= \mathsf{P}\{X(S_n+t)\in\Gamma,\ t<T_n \,|\, X(S_n)=x\}, \quad n\geq 1. \end{aligned}$$

Key Theorem for SRP [8], [11]

A very known Key Renewal Theorem is generalized for SRPr's as follows. Let

(i) The SRPr $X(t), S_n$ be uniformly regular (5),

(ii) The MCh $\{X_n\}$ be non-decomposable, positively recurrent MCh and $\alpha = \{a(x)\ x \in E\}$ is its invariant measure,

(iii) $g(y, t)$ be summing with respect α and integrable on $R^+ = [0, \infty)$,

(iv) Function $Q(x, [t, \infty), E)$ is absolutely continuous at least for one state x positive α-measure.

(v) $m = \sum a(x) \int Q(x, [t, \infty), E) dt$ s a stationary mean of RP length

Then

$$\lim_{t\to\infty} H \star g(t, x) \equiv \lim_{t\to\infty} \int_0^t \sum_{y\in E} H(x, du, y) g(y, t-u) dt = $$

$$= m^{-1} \int_0^\infty \sum_{y\in E} \alpha(y) g(y, t)\, dt.$$

The last statement provides calculation of SRPr stationary states probabilities

$$\pi(\Gamma) = \lim_{t\to\infty} \pi(x, t, \Gamma) = m^{-1} \int_0^\infty \sum_{x\in E} \alpha(x)\pi^{(1)}(x, t, \Gamma)\, dt \qquad (7)$$

in terms of its distributions at separate RP's and invariant measure of MCh.

Remark

The Generalization to the case of general RS space see in [Jacod (1971) [5], Nummelin (1978) [9]].

However, if the behavior of the process is **enough complex** then the calculation of the function $\pi^{(1)}(x, t, \Gamma)$ is also not a simple problem. Its solution can be simplified if at any RP $(S_n^{(1)}, S_{n+1}^{(1)}]$ one can find another **embedded** RT's $S_{n,k}^{(2)}$.

Decomposable semi-regenerative processes

The next development of Smith's regenerative idea has been proposed by V. Rykov, 1975 [10], V.Rykov, S. Jolkov, 1981 [11] by introduction of the nation **Decomposable Semi-Regenerative Process** (DSRP).

For a random interval $T = [\,\underline{S}, \overline{S}\,)$ denote by

$$\mathcal{F}_t^T = \sigma\left\{X(v),\ \underline{S} \le v \le t \wedge \overline{S}\right\}$$

the flow of σ-algebras of events, associated with the process $\{X(t)\}$ behavior at the interval T.

Definition

([Jolkoff, Rykov (1981) [11], Rykov (1997)])
$\mathcal{F}_t^T$ –MT S is called an **Embedded Regeneration Time** (ERT) in the interval T for the process $\{X(t)\}$, if $S \in T$ a.s. and for any $\Gamma \subset E$ the following relation holds

$$\mathrm{P}\left\{X(S+t) \in \Gamma \mid \mathcal{F}_S^T\right\} = \mathrm{P}\left\{X(S+t) \in \Gamma \mid X(S)\right\}.$$

Analogously to SRPr it is possible to show the existence a sequence of such RT's.

Structure of a ERP

In the figure 3 the structure of an ERT at an interval of some RPr are shown.

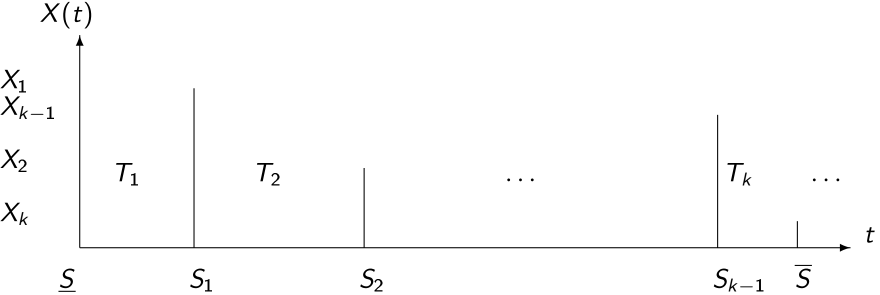

Figure 3: Structure of embedded regeneration periods

Definition

Consider SP $X(t)$ with the sequence of the first level RP's $T_n^{(1)}$ and the embedded RT's of the second level $S_{n,k}^{(2)}$

Definition

With fixed n the triplet $\left\{X(t), S_{n,k}^{(2)}, T_n^{(1)}\right\}$ is called

- the **Embedded Semi-Regenerative Process** (ESRPr), if the relation, analogous to (4), holds for all k and all $\Gamma \subset E$

$$\begin{aligned} & \mathrm{P}\left\{X\left(S_{n,k}^{(2)}+t\right) \in \Gamma \mid \mathcal{F}_{S_{n,k}^{(2)}}^{T_n^{(1)}}\right\} = \mathrm{P}\left\{X\left(S_{n,k}^{(2)}+t\right) \in \Gamma \mid X\left(S_{n,k}^{(2)}\right)\right\} = \\ = \quad & \mathrm{P}\left\{X\left(S_{n,1}^{(2)}+t\right) \in \Gamma \mid X\left(S_{n,1}^{(2)}\right)\right\}. \end{aligned} \tag{8}$$

- It is called the **Decomposable Semi-Regenerative Process** (DSRPr) (of level 2) if it holds for all n;

At that

- the r.v. $S_{nk}^{(2)}$ are called **Embedded Regeneration Times** (ERT's)
- the intervals $T_{nk}^{(2)} = \left(S_{nk}^{(2)},\ S_{nk+1}^{(2)}\right]$ and their lengths $T_{nk}^{(1)} = S_{nk+1}^{(2)} - S_{nk}^{(2)}$ are called **Embedded Regeneration Periods** (ERP), and
- random elements $X_{nk}^{(2)} = X\left(S_{nk}^{(2)}\right)$ are called **Embedded Regeneration States** (ERS).
- The maximal value of the decomposition level of a DSRP is called its **Rank** r.
- The set of regeneration states of the k-th level denote by E_k.

Remarks

Remark

Considering only homogeneous processes let us simplify a little bit notation by omitting one sub-index, such that the super index denote the level of embedded process, and sub-index denote the number of ERT in the upper ERP. Under this remarks the notations are:

$$S_n^{(k)}, \quad T_n^{(k)}, \quad X_n^{(k)}.$$

Remark

Generally speaking, the **embedded regeneration times** *of some process are not its* **regeneration times in the original sense**, *but since their formal definitions are the same, the primary results for the SRP are easily extended to the case of DSRP.*

While analyzing the DSRPr, the role of the ordinary renewal process is played by the **Embedded Renewal Process** (ERP), which for, say k-th level and for $y \in E_k$, is given by

$$N^{(k)}(t, y) = \sum_{n \geq 0} 1_{\{[0,t),y\}} \left(S_n^{(k)}, X_n^{(k)} \right) 1_{\{S_n^{(k)} < T^{(k-1)}\}}.$$

Appropriate **Embedded Renewal Matrix** (ERM) is given by

$$H^{(k)}(x, t, y) = \mathsf{E}_x \left[\sum_{n \geq 0} 1_{\{[0,\, t),\, y\}} \left(S_n^{(k)}, X_n^{(k)} \right) 1_{\{S_n^{(k)} < T^{(k-1)}\}} \right].$$

Remark

Remark

These notions could be generalize for general measurable (not only denumerable) regeneration states space $(E, \mathcal{E})$ *for any* $C \in \mathcal{E}$ *by the following way*

$$N^{(k)}(t, C) = \sum_{n \geq 0} 1_{\{[0,t),C\}} \left(S_n^{(k)}, X_n^{(k)}\right) 1_{\{S_n^{(k)} < T^{(k-1)}\}}.$$

Instead of ERM the appropriate **Embedded Renewal Kernel** *(ERK) should be used that is given by*

$$H^{(k)}(x, t, C) = \mathsf{E}_x \left[\sum_{n \geq 0} 1_{\{[0,\, t),\, C\}} \left(S_n^{(k)}, X_n^{(k)}\right) 1_{\{S_n^{(k)} < T^{(k-1)}\}} \right].$$

Theorem

For ERM the following statement holds.

Theorem

ERK satisfies the following system of equation

$$
\begin{aligned}
& H^{(k)}(x,t,y) = Q_1^{(k)}(x,t,y) + \\
+ & \sum_{j\in E^{(k)}} \int_0^t H^{(k)}(x,du,j)Q^{(k)}(j,t-u,y) - Q^{(k-1)}(x,t,y) = \\
= & Q_1^{(k)}(x,t,y) + H^{(k)} \star Q^{(k)}(x,t,y) - Q^{(k-1)}(x,t,y), \qquad (9)
\end{aligned}
$$

where $Q^{(k)}(x;\ t,\ y)$ and $Q^{(k-1)}$ are the Semi-Markov Matrices of the external and internal embedded periods and the symbol "$\star$" denote matrix convolution, given by this formula.

Proof.

for denumerable states space see in [Jolkoff, Rykov (1981), Rykov (1997)]. For general case it could be done analogously.

Similarly to (6) different characteristics of the DSRP of the $(k-1)$-th level can be expressed in terms of its corresponding characteristics of the k-th level. Particularly, for the one-dimensional distributions

$$\pi^{(k)}(x,t,\Gamma) = \mathrm{P}\left\{X\left(S_{n-1}^{(k)}+t\right)\in\Gamma,\ t<T_n^{(k)} \mid X_{n-1}^{(k)}=x\right\} \quad (k=\overline{1,r})$$

the following representation holds

$$\pi^{(k-1)}(x,t,\Gamma) = H^{(k)} \star \pi^{(k)}(x,t,\Gamma),$$

where ERM $H^{(k)}(x,t,y)$ satisfies the equation (9). These relations make it possible to recover the process distribution by its distribution on separate minimal periods of embedded regeneration.

The limit theorem for SRPr's allows to calculate its stationary distributions, and the system of embedded regeneration periods give it possible to calculate them in terms of distributions at smallest regeneration periods.

Applications

The proposed methods has been applied for different models study

- For the **Priority ququeing systems** investigation (V. Rykov, 1975 [10])
- For the **single -server queue with arbitrary inter-arrival and service time distributions** study (V. Rykov, 1983, 1984 [12] [13]),
- For the complex hierarhical systems study (V. Rykov, 1996 [14], V.Rykov 1997a,b [15, 16]).
- For the polling systems study (V. Rykov, 2009 [17])
- For the reliability of different systems study V. Rykov, D. Kozyrev, 2010 [18]).

In the second part of the talk a new application of the DSRP will be proposed.

On the reliability of a double redundant renewable system with arbitrary distributed life- and repair times of its components

This part of the talk is devoted to one new application of the DSRPr.

The reliability of a double redundant renewable system with arbitrary distributed life- and repair times of its components is investigated. Time dependent and stationary characteristics of the system with the help of a special transformation are founded in closed form.

Introduction and Motivation I

- An investigation of reliability of systems with arbitrary distributions of its components life- and repair times is interesting both from theoretical and practical points of view.
- From theoretical point of view these investigations serve to introduction and development of new mathematical methods.
- From practical point of view they create the base for reliability analysis of different complex systems with arbitrary distributed life- and repair times of their components.
- An investigation of asymptotic behavior of a double cold redundant renewable system under quick restoration has been proposed by Gnedenko and Soloivjev [19], [20], [21].

Introduction and Motivation II

- The investigation of analogous system when one of the input distributions (life- or repair times) has an exponential distribution has been done in V. Rykov V., Tran Ahn Ngia [22], D. Efrosinin, V. Rykov [23].
- Then investigations of more complex systems including those with heterogeneous components and using the simulation methods has been proposed in [24] [25].
- An application of these models one can find in D. Efrosinin, V. Rykov, V. Vishnevskiy [26]
- An investigation of the model with dependent components one can find in V. Rykov, D. Kozyrev D. [27] and D. Kozyrev, N. Kolev, V. Rykov [28]
- The review of these investigations and some of their generalization see in V. Rykov (2018) [29].
- Study of the systems with dependent components was done in [28], [30].

The problem set

In this paper the investigation of a homogeneous cold double redundant system with generally distributed both life- and repair times distributions is considered. Due to special integral transformation, the closed form representations of the main reliability characteristics of such system has been represented. The investigation is based on the stochastic equations for considered in the paper random variables (r.v.'s) and the theory of regenerative processes due to Smith [2].

The problem set. Assumptions and notations I

Consider an homogeneous cold double redundant renewable system with generally distributed life- and repair times, which will be denoted as $< GI_2/GI/1 >$. The system fails when both its components fail. Denote by

- A_i lifetimes of its components
- B_i their repair times.
- It is assumed that the successive lifetimes of the same and different components as well as all their repair times are i.i.d. r.v.'s.
- $$A(t) = \mathrm{P}\{A_i \le t\} \quad B(t) = \mathrm{P}\{B_i \le t\}, \quad A(+0) = B(+0) = 0$$
- $$a = \int_0^\infty (1 - A(x))dx < \infty, \quad b = \int_0^\infty (1 - B(x))dx < \infty.$$
- $E = \{i = 0, 1, 2\}$ the system set of states, i is number of failed components
- $$J(t) = \text{number of failed components in time } t.$$

The problem set. Assumptions and notations II

We are interesting in

- T time to the system failures after its repair.
- reliability function

$$R(t) = \mathsf{P}\{T > t\},$$

- time-dependent state probabilities

$$\pi_j(t) = \mathsf{P}\{J(t) = j\},$$

- stationary probabilities

$$\pi_j = \lim_{t\to\infty} \pi_j(t) \equiv \lim_{t\to\infty} \mathsf{P}\{J(t) = j\},$$

- availability coefficient

$$K_{\text{av.}} = \pi_0 + \pi_1.$$

The problem set. Assumptions and notations III

Further the following notations will be used.

- $$\tilde{a}(s) = \int\limits_0^\infty e^{-sx} dA(x), \quad \tilde{b}(s) = \int\limits_0^\infty e^{-sx} dB(x),$$

- $$\tilde{a}_B(s) = \int_0^\infty e^{-sx} B(x) dA(x), \quad \tilde{b}_A(s) = \int_0^\infty e^{-sx} A(x) dB(x), \tag{10}$$

- $$a_B = -\frac{d}{ds}\tilde{a}_B(s)\Big|_{s=0}, b_A = -\frac{d}{ds}\tilde{b}_A(s)\Big|_{s=0}. \tag{11}$$

- $$\tilde{a}_B(0) = \mathrm{P}\{B \leq A\} \equiv p, \tilde{b}_A(0) = \mathrm{P}\{B > A\} \equiv q = 1 - p.$$

- $$\tilde{a}_B(s) + \tilde{a}_{1-B}(s) = \tilde{a}(s), \quad \tilde{b}_A(s) + \tilde{b}_{1-A}(s) = \tilde{b}(s)$$

- $$\tilde{a}_B(s) + \tilde{b}_A(s) = s \int\limits_0^\infty e^{-sx} A(x) B(x) dx$$

Reliability function I

presented Process J is a regenerative one with delay and its trajectory is

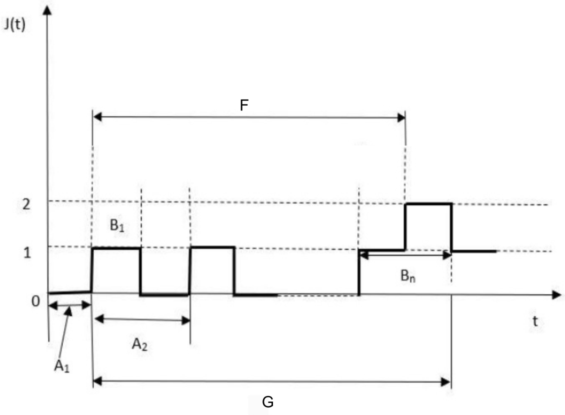

Figure 4: Trajectory of the process J.

Reliability function II

The regeneration times of the process J are:

$$S_0 = 0,\ S_1 = A_1,\ S_2 = S_1 + G_1,\ \ldots,\ S_{k+1} = S_k + G_k,\ \ldots,$$

where regeneration period G is the time interval between two successive returns of the process J into the state 1 after the system failure and repair, when one of the system's components begins operate and the other begins to be repaired. This r.v. satisfies to the equation

$$G = \begin{cases} A + G & \text{if } B < A, \\ B & \text{if } B > A. \end{cases} \tag{12}$$

Denote by F the time between failures and by F_1 the time to the first system failure 4. From it one can see that the time between the system failures F satisfies to the stochastic equation

$$F = \begin{cases} A + F & \text{if } B < A, \\ A & \text{if } B > A, \end{cases} \tag{13}$$

and the time to the first system failure equals to:

$$F_1 = A + F. \tag{14}$$

The c.d.f.'s of these r.v.'s denote by

$$G(t) = \mathsf{P}\{G \le t\}, \quad F(t) = \mathsf{P}\{F \le t\}, \quad F_1(t) = \mathsf{P}\{F_1 \le t\}$$

and their m.g.f.'s by $\tilde{g}(s)$, $\tilde{f}(s)$, $\tilde{f}_1(s)$.

Applying Laplace transform to stochastic equation (12) one can find the equation

$$\begin{aligned}
\tilde{g}(s) &= \mathsf{E}\left[e^{-sG}\right] = \int\limits_0^\infty e^{-st} dG(t) \\
&= \int\limits_0^\infty dA(x)\left[B(x)e^{-sx}\tilde{g}(s) + \int_{y>x} e^{-sy} dB(y)\right] = \\
&= \tilde{g}(s)\int\limits_0^\infty e^{-sx}B(x)dA(x) + \int_0^\infty e^{-sy}A(y)dB(y) = \\
&= \tilde{g}(s)\tilde{a}_B(s) + \tilde{b}_A(s),
\end{aligned}$$

from which the expression for the regeneration period m.g.f. follows:

$$\tilde{g}(s) = \mathsf{E}[e^{-sG}] = \frac{\tilde{b}_A(s)}{1 - \tilde{a}_B(s)}. \tag{15}$$

Analogously from the equality (13) one can find

$$\begin{aligned}\tilde{f}(s) &= \mathsf{E}\left[e^{-sT}\right] = \int_0^\infty e^{-st}dF(t) = \\ &= \int_0^\infty e^{-sx}\left[B(x)\tilde{f}(s) + (1-B(x))\right]dA(x) = \\ &= \tilde{f}(s)\tilde{a}_B(s) + \tilde{a}_{1-B}(s),\end{aligned}$$

from which the expression for m.g.f.' of the system times between failures and time to the first failure holds:

$$\tilde{f}(s) = \frac{\tilde{a}_{1-B}(s)}{1-\tilde{a}_B(s)}, \quad \tilde{f}_1(s) = \frac{\tilde{a}(s)\tilde{a}_{1-B}(s)}{1-\tilde{a}_B(s)}. \tag{16}$$

In these equations the notations (10, 11) are used.

From the expression (15) one can firnd the mean of the regeneration period,

$$g = \mathrm{E}[G] = -\tilde{g}'(0) = \frac{a_B + b_A}{q}, \tag{17}$$

Taking into account the expression

$$\tilde{F}(s) = \int_0^\infty e^{-st} F(t) dt = \frac{1}{s} \int_0^\infty e^{-st} dF(t) = \frac{1}{s} \tilde{f}(s)$$

for LT $\tilde{R}(s)$ of the reliability function $R(t) = 1 - F(t)$ from (16) it holds

$$\tilde{R}(s) = \frac{1}{s} - \tilde{F}(s) = \frac{1}{s}(1 - \tilde{f}(s)) = \frac{1 - \tilde{a}_B(s) - \tilde{a}_{1-B}(s)}{s(1 - \tilde{a}_B(s))} = \frac{1 - \tilde{a}(s)}{s(1 - \tilde{a}_B(s))}. \tag{18}$$

At that for the mean system life time it follows:

$$\mathrm{E}[F] = \tilde{R}(0) = \frac{a}{1 - \tilde{a}_B(0)} = \frac{a}{q}. \tag{19}$$

System state probabilities

For the system state probabilities calculation we use the theory of regenerative processes, accordingly to which the state probabilities of the process in any time t

$$\pi_j(t) = \mathsf{P}\{J(t) = j\} \quad (j = 0, 1, 2)$$

can be represented in terms of its distribution on separate regeneration period

$$\pi_j^{(1)}(t) = \mathsf{P}\{J(t) = j,\ t < G\} \quad (j = 0, 1, 2)$$

and the renewal function $H(t)$ as follows

$$\pi_i(t) = \pi_i^{(1,1)}(t) + \int_0^\infty dH(u)\pi_j^{(1)}(t-u). \tag{20}$$

In our case as it was mentioned before the process J is a regenerative process with delay (see figure 4).

Thus the the first period differs from the others, has the duration A and if the system functioning begins when both components are in good state the process J distribution on it has the form

$$\pi_j^{(1,1)}(t) = \mathsf{P}\{J(t) = j,\ t < A\} = \delta_{j0}(1 - A(t)), \tag{21}$$

In terms of the c.d.f. $G(t) = \mathsf{P}\{G_i \leq t\}$ of r.v.'s G_i the renewal function $H(t)$ is determined as follows

$$H(t) = \sum_{k\geq 1} \mathsf{P}\left\{\left[\sum_{1\leq i\leq k} G_i\right] \leq t\right\} = \sum_{k\geq 1} G^{*k}(t), \tag{22}$$

Accordingly to the key Smith theorem for regenerative processes [2] the steady state probabilities of the process exists and has the form

$$\lim_{t\to\infty} \pi_j(t) = \frac{1}{\mathsf{E}[G]} \int_0^\infty \pi_j^{(1)}(t)dt = \frac{1}{\mathsf{E}[G]} \int_0^\infty \mathsf{E}[1_{\{J(t)=j\}}]dt = \frac{\mathsf{E}[G_j]}{\mathsf{E}[G]}, \tag{23}$$

where G_i is the time spent by the system in its state i, $(i \in E)$ during regeneration time G. For calculation these values ant their expectations consider the process J behavior on a separate regeneration period.

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Investigation of the system behavior on a separate regeneration period

Consider the system behavior on a separate regeneration period. Remind that G denoted the length of regeneration period, and G_j is the time spent by the process J in its states $j = \{0, 1, 2\}$ on its. As it can be seen from the figure 4 they satisfies to the following stochastic equations:

$$
\begin{aligned}
G_0 &= \begin{cases} 0 & \text{if } A \leq B, \\ A - B + G_0 & \text{if } A > B; \end{cases} \\
G_1 &= \begin{cases} A & \text{if } A \leq B, \\ B + G_1 & \text{if } A > B; \end{cases} \\
G_2 &= \begin{cases} B - A & \text{if } A \leq B, \\ G_2 & \text{if } A > B. \end{cases}
\end{aligned} \tag{24}
$$

Calculating the expectations of r.v.'s G_i $(i = 0, 1, 2,)$ form (24) one can find the equations:

$$
\begin{aligned}
g_0 \equiv \mathrm{E}[G_0] &= \int_{x\geq 0}\int_{y\leq x}(x - y + g_0)dA(x)dB(y) = \\
&= pg_0 + \int_{x\geq 0}[xB(x) - \int_{y\leq x} ydB(y)]dA(x) = pg_0 + a_B - b + b_a, \\
g_1 \equiv \mathrm{E}[G_1] &= \int_{x\geq 0} x(1 - A(x))dB(x) + \int_{x\geq 0}(x + g_1)B(x))dA(x) = \\
&= a + b - (a_B + b_A) + g_1 p, \\
g_2 \equiv \mathrm{E}[G_2] &= \int_{x\geq 0}\int_{y\leq x}(x - y)dA(y)dB(x) + g_2 p = \\
&= pg_2 + \int\limits_{x\geq 0}[xA(x) - \int\limits_{y\geq x} ydA(y)]dB(x) = \\
&= pg_2 + b_A - \int\limits_{y\geq 0} ydA(y)(1 - B(y)) = pg_2 + b_A - a + A_B.
\end{aligned}
$$

Thus it holds

$$g_0 = \frac{a_b + b_A - b}{q}, \quad g_1 = \frac{(a+b) - (a_b + b_A)}{q}, \quad g_2 = \frac{a_B + b_A - a}{q},$$

and therefore taking into account the expression (17) for E[G] the stationary probabilities are:

$$\pi_0 = 1 - \frac{b}{a_B + b_A} \quad \pi_1 = \frac{a+b}{a_B + b_A} - 1, \quad \pi_2 = 1 - \frac{a}{a_B + b_A}. \tag{25}$$

Conclusion and Acknowledgments

With the help of embedded regeneration theory and special transformations the closed form representation of the reliability function and the steady state probabilities for the double redundant renewable system with generally distributed life- and repair times of its components are found.

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