QUEUING THEORY SCHOOL

21-26 april

Mathematical Models of Queuing-Inventory Systems with Catastrophes

Prof Dr Agassi Melikov, Laman Poladova

Baku Engineering University

agassi.melikov@gmail.com

amelikov@beu.edu.az

Outline

- Introduction
- Mathematical Models of İnfinite Queuing-Inventory Systems Under (s, S) Policy
- Mathematical Models of İnfinite Queuing-Inventory Systems Under Randomized Policy
- Mathematical Models of İnfinite Queuing-Inventory Systems Under (s, Q) Policy
- Mathematical Models of İnfinite Queuing-Inventory Systems Under Base Stock Policy
- Mathematical Models of Finite Queuing-Inventory Systems Under (s, S) Policy

Introduction

- Note that the first papers devoted to models of QISs were published by Sigman and Simchi-Levi [1] and Melikov and Molchanov [2] independently of each other. For detailed review see Krishnamoorthy, Shajin, and Narayanan [3].

1. Melikov, A.; Molchanov, A. Stock Optimization in Transport/Storage Systems. *Cybernetics* **1992**, *28*, 484–487.
2. Sigman, K.; Simchi-Levi, D. Light Traffic Heuristic for an M/G/1 Queue with Limited Inventory. *Ann. Oper. Res.* **1992**, *40*, 371–380. [CrossRef]
3. Krishnamoorthy, A.; Shajin, D.; Narayanan, W. *Inventory with Positive Service Time: A Survey, Advanced Trends in Queueing Theory*; Series of Books "Mathematics and Statistics" Sciences. V., 2; Anisimov, V., Limnios, N., Eds.; ISTE & Wiley: London, UK, 2021; pp. 201–238.

For the first time the IMSs with positive service times were called queuing-inventory systems (QIS) in the papers Schwarz and Daduna [4] and Schwarz et al. [5].

4. Schwarz, M.; Daduna, H. Queuing Systems with Inventory Management with Random Lead Times and with Backordering. *Math. Methods Oper. Res.* **2006**, 64, 383–414. [CrossRef]

5. Schwarz, M.; Sauer, C.; Daduna, H.; Kulik, R.; Szekli, R. M/M/1 Queuing Systems with Inventory. *Queuing Syst. Theory Appl.* **2006**, *54*, 55–78. [CrossRef]

Common RPs

(1) (s, S)-policy: in this policy, the replenishment size is that much to bring the level back to Sat the replenishment epoch, where s is the recorded level and S is the maximum capacity of the warehouse; sometimes this policy is called "Up to S" policy.

(2) (s, Q)-policy: in this policy, the replenishment size is fixed and is equal to Q=S-s; in this policy to avoid repeated replenishment, it is assumed that $s<(S/2)$.

(3) Randomized policy: in this policy, the probability that replenishment size is n equal to p_n, such that $\sum_{n=1}^{S} p_n = 1$, where $p_S > 0$.

(4) Base stock policy: in this policy, a replenishment is called every time an item sells out; sometimes this RP is called either (S-1, S)-policy or one-to-one ordering policy. This policy is advised for bulky, expensive items with low demand and slow lead times.

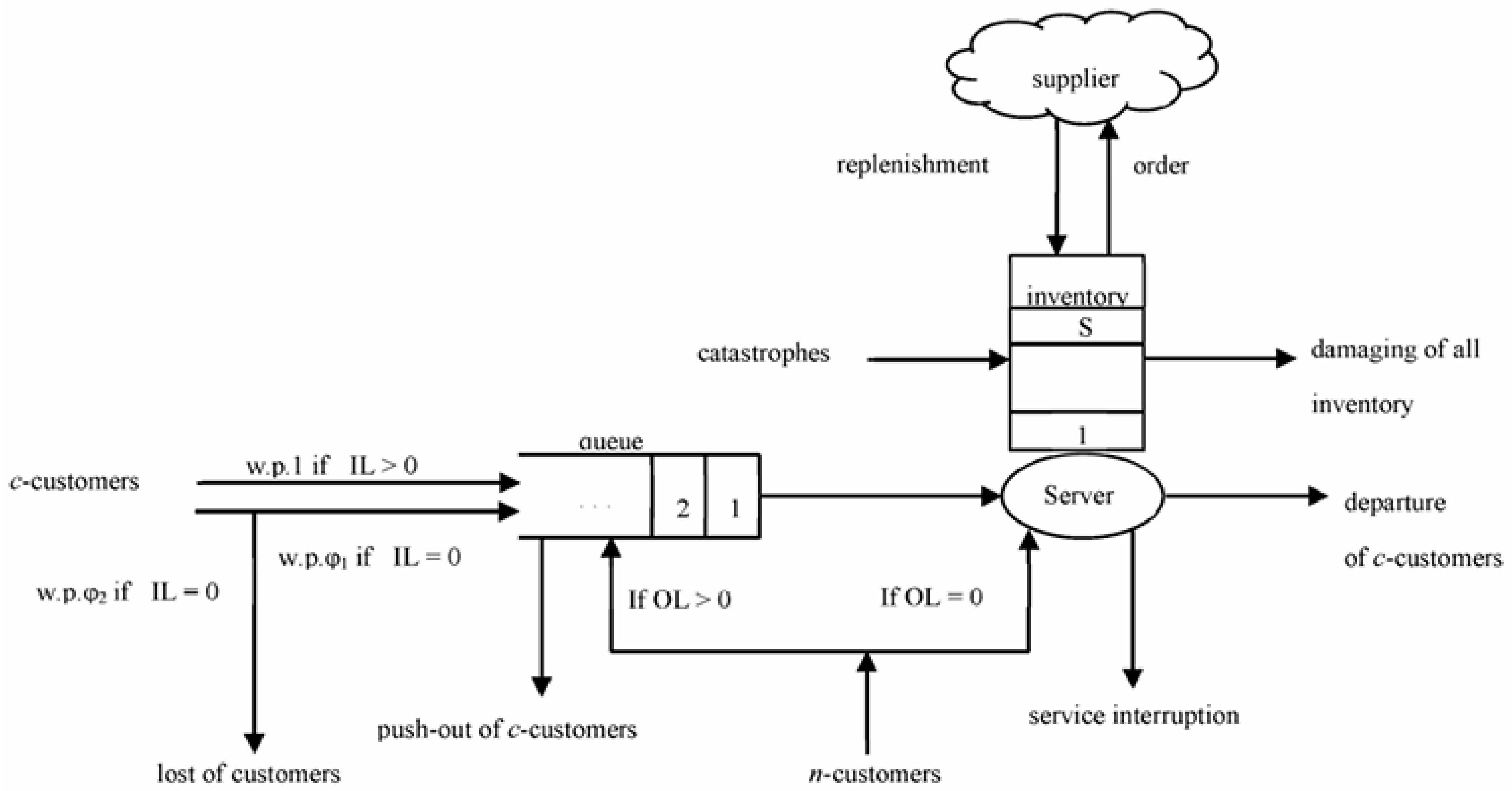


Figure 1. Block diagram of the system under study.

The block diagram of the investigated single-server QIS of infinite capacity is shown in Figure 1. The homogeneous c-customers arrive at the service facility according to Poisson process with rate λ^+. The service times of the c-customers are assumed to be exponentially distributed with parameter μ. The service requires an idle server along with items (one for each c-customer) that are stored in an inventory of maximum capacity S.

In the system, hybrid sales scheme is used, i.e., some part of c-customers is serviced according to the backorder sale scheme, while the other part is serviced according to the lost sale scheme. This means the following: if there are no stocks in the system upon arrival of c-customer, then, in accordance to the Bernoulli trials, it either, with probability (w.p.), φ_1 joins the queue of infinite length (backorder sale scheme), or w.p. φ_2 leaves the system unserved (lost sale scheme), where $\varphi_1 + \varphi_2 = 1$.

The system also receives n-customers with a rate λ^-. When a n-customer arrives, one c-customer force out of the system. A n-customer can force out of the system even a c-customer, which is in the server, while the inventory level does not change, since it is assumed that stocks are released after the completion of servicing a c-customer. If there is a queue of c-customers at the time an n-customer arrives, then only the c-customer is pushed out from the queue (i.e., the service of the c-customer, which is in the server, continues); if there are no c-customers in the system, then the received n-customer does not affect the operation of the system.

In the system, catastrophic events can occur only in its warehouse part. The flow of catastrophic events is Poisson one with the parameter κ, and at the moment of arrival of such an event, all the reserves of the system are instantly destroyed. As a result of the catastrophes, even the stock, which is at the status of release to the c-customer, is destroyed. In the latter case, the c-customer whose service was interrupted due to a catastrophe is returned to the queue; in other words, the catastrophe only destroys the stocks of the system and does not force c-customers out of the system. If the inventory level is zero, then the disaster does not affect the operation of the system warehouse.

Here, two inventory replenishment policies were considered. The first RP was according to a (s, S)-type policy (sometimes this policy is called "Up to S"). In this policy, when the inventory level drops to the reorder point s, where $0 \le s < S$, an order was placed for replenishment and upon replenishment, the inventory level was restocked to level S, no matter how many items are still present in the inventory. Second RP is randomized (randomized replenishment policy, RRP). In RRP, an order is placed only when the system's warehouse is completely empty and the volume of the supplied stock is a random variable with a known distribution; in other words, w.p. α_m, the volume of incoming stock is equal to m, where $\sum_{m=1}^{S} \alpha_m = 1,\ \alpha_S > 0$. In both RPs, the parameter ν indicates the reorder rate per order.

The task is to find the joint distribution of the number of c-customers in the system and the inventory level of the system, as well as to calculate the key performance measures of the system.

Model Under (s,S) Policy

Let X_t be the number of customers at time t and Y_t be the inventory level

at time t. Then, the process $Z_t = \{(X_t, Y_t), t \geq 0\}$ forms a continuous time Markov chain (CTMC) with state space

$$E = \bigcup_{n=0}^{\infty} L(n),$$

where $L(n) = \{(n,0),(n,1),\ldots,(n,S)\}$ is the subset of state space E with $X_t = n$ called the level n.

Let $q((n_1,m_1),(n_2,m_2))$ denote the transition rate from state $(n_1,m_1) \in E$ to state $(n_2,m_2) \in E$.

investigated CTMC has a generator $G = (q((n_1,m_1),(n_2,m_2))),\ (n_1,m_1),(n_2,m_2) \in E$, with the following transition rates for $(n_1,m_1) \in E$:

$$q((n_1, m_1), (n_1 + 1, 0)) = \lambda^+ \varphi_1 \cdot \chi(m_1 = 0); \tag{1}$$

$$q((n_1, m_1), (n_1 + 1, m_1)) = \lambda^+ \cdot \chi(m_1 > 0); \tag{2}$$

$$q((n_1, m_1), (n_1 - 1, m_1)) = \lambda^- \cdot \chi(n_1 > 0); \tag{3}$$

$$q((n_1, m_1), (n_1 - 1, m_1 - 1)) = \mu \cdot \chi(n_1 > 0) \cdot \chi(m_1 > 0); \tag{4}$$

$$q((n_1, m_1), (n_1, 0)) = \kappa \cdot \chi(m_1 > 0); \tag{5}$$

$$q((n_1, m_1), (n_1, S)) = \nu \cdot \chi(m_1 \leq s). \tag{6}$$

Hereinafter, $\chi(A)$ is the indicator function of the event A, which is 1 if A is true and 0 otherwise.

By re-numbering the states of the system in a lexicographic way, from relations (1)–(6) we conclude that the process $Z_t, t \geq 0$, is a level independent quasi birth–death (LIQBD) process and its generator G might be represented as follows:

$$G = \begin{pmatrix} B & A_0 & O & \dots & O & \dots \\ A_2 & A_1 & A_0 & \cdots & O & \dots \\ O & A_2 & A_1 & A_0 & O & \dots \\ O & O & A_2 & A_1 & A_0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}, \tag{7}$$

where O denotes zero square matrix with dimension $S+1$, and all other block matrices are square matrices of the same dimension. Entities of the block matrices $B = \|b_{ij}\|$ and $A_k = \|a_{ij}^{(k)}\|, i, j = 0, 1, \dots, S$, are determined as follows:

$$b_{ij} = \begin{cases} \nu & \text{if } 0 \leq i \leq \text{s}, j = S, \\ \kappa & \text{if } i > 0, j = 0, \\ -(\nu + \lambda^+ \varphi_1) & \text{if } i = j = 0, \\ -(\nu + \kappa + \lambda^+) & \text{if } 0 < i \leq s, i = j, \\ -(\kappa + \lambda^+) & \text{if } s < i \leq S, i = j, \\ 0 & \text{in other cases;} \end{cases} \tag{8}$$

$$a_{ij}^{(0)} = \begin{cases} \lambda^+ \varphi_1 & \text{if } i = j = 0, \\ \lambda^+ & \text{if } i \neq 0, i = j, \\ 0 & \text{in other cases;} \end{cases} \tag{9}$$

$$a_{ij}^{(1)} = \begin{cases} \nu & \text{if } 0 \leq i \leq s, j = S, \\ \kappa & \text{if } i > 0, j = 0, \\ -(\lambda^- + \nu + \lambda^+ \varphi_1) & \text{if } i = j = 0, \\ -(\nu + \kappa + \mu + \lambda^+ + \lambda^-) & \text{if } 0 < i \leq s, i = j, \\ -(\kappa + \mu + \lambda^+ + \lambda^-) & \text{if } i > s, i = j, \\ 0 & \text{in other cases;} \end{cases} \tag{10}$$

$$a_{ij}^{(2)} = \begin{cases} \lambda^- & \text{if } i = j, \\ \mu & \text{if } i > 0, j = i - 1, \\ 0 & \text{in other cases.} \end{cases} \tag{11}$$

The entities of the generator $A = A_0 + A_1 + A_2$ are determined as follows:

$$a_{ij} = \begin{cases} -\nu & \text{if } i = j = 0, \\ \nu & \text{if } 0 \leq i \leq s, j = S, \\ \mu + \kappa & \text{if } i = 1, j = 0, \\ \kappa & \text{if } i > 1, j = 0, \\ -\mu & \text{if } i > 0, j = i, \\ \mu & \text{if } i \geq 2, j = i - 1. \end{cases} \tag{12}$$

The stationary probability vector that corresponds to the generator A is denoted by $\boldsymbol{\pi} = (\pi(0), \pi(1), \ldots, \pi(S))$. In other words, we have the balance equations:

$$\boldsymbol{\pi} A = 0, \boldsymbol{\pi} \boldsymbol{e} = 1, \tag{13}$$

where $\mathbf{0}$ is the null row vector of dimension $S + 1$ and $\boldsymbol{e}$ is the column vector of dimension $S + 1$ that contains only 1's.

By using the recursive procedure, we obtained that Equation (13) had the following solution:

$$\pi(0) = \frac{1+bc}{1+dc}, \pi(1) = d\pi(0) - b; \pi(m) = a_m \pi(1), 2 \leq m \leq S, \tag{14}$$

where $d = \frac{\nu+\kappa}{\mu}$, $b = \frac{\kappa}{\mu}$, $c = \sum_{m=1}^{S} a_m$, $a_m = \begin{cases} (1+d)^{m-1}, & \text{if } 1 \leq m \leq s+1, \\ (1+d)^s (1+b)^{m-s-1}, & \text{if } s+1 < m \leq S. \end{cases}$

Using the stationary probability vector of the generator A given by (14), we can derive the ergodicity (stability) condition of the process $Z_t, t \geq 0$.

Proposition 1. *Under (s, S) policy, the process $Z_t, t \geq 0$, is ergodic if and only if the following condition is fulfilled:*

$$\lambda^+(1 - \varphi_2 \pi(0)) < \lambda^- + \mu(1 - \pi(0)). \tag{15}$$

Proof of Proposition 1. In accordance with Neuts, the process $Z_t, t \geq 0$, is ergodic if and only if

$$\pi A_0 e < \pi A_2 e. \tag{16}$$

Special Cases

Note 1. *The established ergodicity condition (15) has a probabilistic meaning, i.e., it indicates that the rate of c-customers entering the system must be less than the total rate of negative customers and the rate of served c-customers. We find from (15) that in general case stability condition for the present model is dependent on the storage size of system, the rate of catastrophes, and the replenishment rate.*

Note 2. *Consider the following special cases.*

(i) If $\varphi_2 = 1$ (i.e., when a pure lost sale scheme is used) and $\lambda^- = 0$ (i.e., when there are not negative customers) from (9), we find the ergodicity condition for the single-server Markovian queuing system, i.e., $\lambda^+ < \mu$. In other words, under such assumptions, the ergodicity condition of the system does not depend on the storage size of system, the rate of catastrophes, and the replenishment rate. Similar results for other models were obtained in Krishnamoorthy and his students

(ii) If $\varphi_2 = 1$ and $\lambda^- > 0$, the ergodicity condition is depending on all indicated parameters of the system, see Formula (14).

Calculations of SSPs

A steady-state probability that corresponds to the generator matrix G, we denote by $p = (p_0, p_1, p_2, \cdots)$, where $p_n = (p(n,0), p(n,1), \ldots, p(n,S)), n = 0, 1, \cdots$. Under the ergodicity condition (15), desired steady-state probabilities are determined from the following equations:

$$p_n = p_0 R^n, n \geq 1, \tag{17}$$

where R is the nonnegative minimal solution of the following quadratic matrix equation:

$$R^2 A_2 + R A_1 + A_0 = 0.$$

From (8)–(11), it was concluded that bound probabilities p_0 are determined from the following system of equations with normalizing conditions:

$$p_0(B + R A_2) = 0,$$

$$p_0(I - R)^{-1} e = 1. \tag{18}$$

where I indicate the identity matrix of dimension $S + 1$.

Under RRP

Now consider the computation of the steady-state probabilities under RRP. In this case, parameters $q((n_1, m_1), (n_2, m_2))$ are calculated via relations (1)–(5) but relation (6) should be substituted by the following equations:

$$q((n_1, 0), (n_1, m)) = \nu_m \cdot \chi(1 \le m \le S),$$

where $\nu_m = \nu\alpha_m,\ 1 \le m \le S$.

Therefore, for this policy the generator matrix of the process $Z_t, t \ge 0$, has the following form:

$$\widetilde{G} = \begin{pmatrix} \widetilde{B} & A_0 & O & \dots & O & \dots \\ A_2 & \widetilde{A}_1 & A_0 & \cdots & O & \dots \\ O & A_2 & \widetilde{A}_1 & A_0 & O & \dots \\ O & O & A_2 & \widetilde{A}_1 & A_0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix},$$

Here, entities of matrices $\widetilde{B}$ and $\widetilde{A}_1$ are calculated as follows:

$$\widetilde{b}_{ij} = \begin{cases} \nu_j & \text{if } i = 0, j > 0, \\ \kappa & \text{if } i > 0, j = 0, \\ -(\nu + \lambda^+ \varphi_1) & \text{if } i = j = 0, \\ -(\kappa + \lambda^+) & \text{if } 0 < i \leq S, i = j, \\ 0 & \text{in other cases ;} \end{cases} \tag{19}$$

$$\widetilde{a}_{ij}^{(1)} = \begin{cases} \nu_j & \text{if } i = 0, j > 0, \\ \kappa & \text{if } i > 0, j = 0, \\ -(\lambda^- + \nu + \lambda^+ \varphi_1) & \text{if } i = j = 0, \\ -(\kappa + \mu + \lambda^+ + \lambda^-) & \text{if } i > 0, i = j, \\ 0 & \text{in other cases.} \end{cases} \tag{20}$$

In this model, entities of the generator $\widetilde{A} = A_0 + \widetilde{A}_1 + A_2$ are determined as

$$\widetilde{a}_{ij} = \begin{cases} -\nu & \text{if } i = j = 0, \\ \nu_j & \text{if } i = 0, j > 0, \\ \mu + \kappa & \text{if } i = 1, j = 0, \\ \kappa & \text{if } i > 1, j = 0, \\ -\mu & \text{if } i > 0, j = i, \\ \mu & if\ i \geq 2, j = i - 1. \end{cases} \tag{21}$$

Again, using the recursive procedure, we found that the balance Equation (13), where the matrix A is replaced by $\widetilde{A}$, the following solution was used

$$\pi(m) = r_m \pi(0),\ 0 \le m \le S, \tag{22}$$

where r_m are calculated from the following reverse recursive relations

$$r_0 = 1,$$

$$r_S = \frac{\nu_S}{\mu + \kappa},$$

$$r_m = \frac{1}{\mu + \kappa}(\mu r_{m+1} + \nu_m),\ 1 \le m \le S - 1.$$

Here, the unknown parameter $\pi(0)$ is found from the normalizing condition, i.e.,

$$\pi(0) = \left(\sum_{r=0}^{S} r_m \right)^{-1}. \tag{23}$$

In analogy with Proposition 1, it is easy to show that the following fact is true.

Proposition 2. *Under RRP policy, the process $Z_t, t \geq 0$, is ergodic if and only if the condition (15) is fulfilled where $\pi(0)$ is defined as in (23).*

Furthermore, by using a system of Equations (17) and (18), the steady-state probabilities for this model were calculated.

Performance Measures

In this section, we are interested in the key performance measures of the investigated system related to both inventory and queuing under each RP. Having determined the steady-state probabilities under both RPs, we can compute the key performance measures of the investigated models explicitly.

Performance measures related to inventory are the following:

- Average inventory level (S_{av}) under both policy

$$S_{av} = \sum_{m=1}^{S} m \sum_{n=0}^{\infty} p(n, m); \tag{24}$$

- Average order size under (s, S) policy

$$V_{av} = \sum_{m=S-s}^{S} m \sum_{n=0}^{\infty} p(n, S-m); \tag{25}$$

Performance Measures

under RRP

$$V_{av} = \left(\sum_{m=1}^{S} m\alpha_m\right)\left(\sum_{n=0}^{\infty} p(n,0)\right); \tag{26}$$

- Average reorder rate (RR) under (s, S) policy

$$RR = \mu \sum_{n=1}^{\infty} p(n, s+1) + \kappa\left(1 - \sum_{n=0}^{\infty} p(n,0)\right); \tag{27}$$

under RRP

$$RR = \mu \textstyle\sum_{n=1}^{\infty} p(n,1) + \kappa\left(1 - \sum_{n=0}^{\infty} p(n,0)\right). \tag{28}$$

Performance Measures

Performance measures related to queuing are the following:

- Average length of the queue (L_{av}) under both policies

$$L_{av} = \sum_{n=1}^{\infty} n \sum_{m=0}^{S} p(n,m). \tag{29}$$

- Loss rate (LR) of customers under both policies

$$LR = \lambda^{+}\varphi_2 \sum_{n=0}^{\infty} p(n,0) + \lambda^{-}\left(1 - \sum_{m=0}^{S} p(0,m)\right). \tag{30}$$

Numerical Results

First, consider the results for the model with "Up to S" policy. For this RP, we considered the behavior of performance measures versus s as well as the finding the optimal value of s to minimize the expected total cost (ETC) that was defined as follows:

$$ETC(s) = (K + c_r V_{av})RR + c_h S_{av} + c_{ps} \kappa S_{av} + c_l LR + c_w L_{av}, \tag{31}$$

where K is the fixed price of one order, c_r is the unit price of the order size, c_h is the unit inventory storage price per unit of time, c_{ps} is the price of unit inventory damaging, c_l is the cost for a single c-customer loss, c_w is the price per unit time of queuing delay for a single c-customer.

Numerical Results

For this policy, it was assumed that values of all parameters of the QIS were fixed except the parameter s. In other words, here, numerical experiments were processed to analyze the effect of parameter s on the performance measures.

Let us consider $S = 50$ and that values of load parameters are selected as follows: $\lambda^+ = 6$, $\lambda^- = 1$, $\kappa = 1, \mu = 8, \varphi_1 = 0.6, \nu = 1$. The coefficients in the expression for functional in ETC (see (31)) were chosen as follows: $K = 10, c_r = 15, c_h = 10, c_l = 450$, $c_w = 400, c_{ps} = 15$.

The impact of reorder points s on performance measures, ETC, are shown in Table 1. From this table, we conclude that the rate of change of all performance measures was very low and ETC was a unimodal function; its minimal value is indicated in bold.

For this policy, it was assumed that values of all parameters of the QIS were fixed except the parameter s. In other words, here, numerical experiments were processed to analyze the effect of parameter s on the performance measures.

Let us consider $S = 50$ and that values of load parameters are selected as follows: $\lambda^+ = 6$, $\lambda^- = 1$, $\kappa = 1, \mu = 8, \varphi_1 = 0.6, \nu = 1$. The coefficients in the expression for functional in ETC (see (31)) were chosen as follows: $K = 10, c_r = 15, c_h = 10, c_l = 450$, $c_w = 400, c_{ps} = 15$.

The impact of reorder points s on performance measures, ETC, are shown in Table 1. From this table, we conclude that the rate of change of all performance measures was very low and ETC was a unimodal function; its minimal value is indicated in bold.

Table 1. Impact of reorder point s to performance measures and ETC.

s	S_{av}	V_{av}	L_{av}	RR	LR	ETC
1	21.4427	25.0148	14.1234	0.5004	2.4279	8176.89
2	21.4447	25.0169	14.1208	0.5005	2.4278	8175.77
3	21.4470	25.0193	14.1183	0.5006	2.4277	8174.74
4	21.4495	25.0219	14.1161	0.5008	2.4276	8173.85
5	21.4524	25.0249	14.1142	0.5009	2.4276	8173.12
6	21.4557	25.0282	14.1124	0.5011	2.4275	8172.47
7	21.4593	25.0319	14.1109	0.5013	2.4274	8171.96
8	21.4646	25.0348	14.1097	0.5015	2.4274	8171.64
9	21.4681	25.0407	14.1083	0.5018	2.4274	8171.22
10	21.4732	25.0459	14.1072	0.5021	2.4273	8171.00
11	21.4825	25.0480	14.1061	0.5025	2.4273	8170.92
12	21.4855	25.0583	14.1053	0.5028	2.4273	8170.81
13	21.4948	25.0655	14.1045	0.5032	2.4272	8170.84
14	21.5021	25.0724	14.1039	0.5036	2.4272	8171.02
15	21.5099	25.0827	14.1032	0.5043	2.4272	8171.19
16	21.5200	25.0929	14.1026	0.5049	2.4272	8171.50
17	21.5318	25.1054	14.1020	0.5058	2.4272	8172.04
18	21.5348	25.1166	14.1016	0.5066	2.4272	8172.45
19	21.5577	25.1305	14.1012	0.5076	2.4271	8173.11
20	21.5731	25.1459	14.1009	0.5087	2.4271	8173.99
21	21.5913	25.1618	14.1006	0.5101	2.4271	8174.78
22	21.6091	25.1820	14.1002	0.5116	2.4271	8175.86
23	21.6300	25.2029	14.1000	0.5133	2.4271	8177.14
24	21.6554	25.2233	14.0998	0.5154	2.4271	8178.81
25	21.6785	25.2514	14.0996	0.5177	2.4271	8180.22
26	21.6971	25.2764	14.0994	0.5194	2.4271	8182.75
27	21.7322	25.3014	14.0992	0.5218	2.4271	8184.44
28	21.7708	25.3438	14.0991	0.5271	2.4271	8186.67
29	21.8121	25.3939	14.0989	0.5329	2.4271	8189.58
30	21.8532	25.4310	14.0988	0.5399	2.4271	8192.77

The goals of the numerical experiments for the model with RRP were the investigation of the behavior of performance measures versus initial parameters for three schemas of changing of probabilities α_m, $1 \le m \le S$: (1) when α_m, $1 \le m \le S$ are constants, (2) when α_m, $1 \le m \le S$ are increasing ones, and (3) when α_m, $1 \le m \le S$ are decreasing ones.

Here, we again assumed that $S = 50$ and $\varphi_1 = 0.6$. Additionally, in the first schema, we set $\alpha_m = \frac{1}{50}$, $1 \le m \le 50$; in the second schema, we set $\alpha_1 = 0.01755$, $\alpha_m = \alpha_{m-1} + 0.0001$, $2 \le m \le 50$; in the third schema, we set $\alpha_1 = 0.02245$, $\alpha_m = \alpha_{m-1} - 0.0001$, $2 \le m \le 50$;

Values of other parameters are shown in the title of the appropriate Tables 2–5. In these tables, the first row corresponds to schema (1), the second row corresponds to schema (2), and the third row corresponds to schema (3).

Table 2. Performance measures vs. λ^+ under RRP, $\lambda^- = 1,\ \mu = 15,\ \nu = 1, \kappa = 1$.

λ^+	S_{av}	V_{av}	L_{av}	RR	LR
5	10.4293 10.8777 9.9749	13.6926 13.5965 13.7905	2.7998 2.7612 2.8397	0.5370 0.5332 0.5408	1.7367 1.7260 1.7475
5.2	10.3387 10.7845 9.8870	13.7382 13.6384 13.8400	3.0565 3.0111 3.1036	0.5388 0.5348 0.5427	1.8006 1.7892 1.8122
5.4	10.2490 10.6920 9.7998	13.7845 13.6810 13.8901	3.3380 3.2847 3.3936	0.5406 0.5365 0.5447	1.8645 1.8524 1.8769
5.6	10.1600 10.6003 9.7133	13.8315 13.7243 13.9409	3.6481 3.5854 3.7138	0.5424 0.5382 0.5467	1.9286 1.9156 1.9417
5.8	10.0718 10.5095 9.6276	13.8792 13.7685 13.9923	3.9915 3.9175 4.0692	0.5443 0.5399 0.5487	1.9926 1.9789 2.0067
6	9.9845 10.4197 9.5428	13.9276 13.8134 14.0445	4.3739 4.2864 4.4661	0.5462 0.5417 0.5508	2.0568 2.0423 2.0717
6.2	9.8981 10.3308 9.4588	13.9769 13.8591 14.0975	4.8024 4.6985 4.9122	0.5481 0.5435 0.5528	2.1212 2.1058 2.1370
6.4	9.8127 10.2430 9.3757	14.0269 13.9056 14.1511	5.2859 5.1620 5.4175	0.5501 0.5453 0.5549	2.1858 2.1696 2.2024
6.6	9.7283 10.1562 9.2930	14.0777 13.9530 14.2056	5.8360 5.6875 5.9926	0.5521 0.5472 0.5571	2.2506 2.2335 2.2681
6.8	9.6450 10.0706 9.2125	14.1294 14.0013 14.2608	6.4678 6.2886 6.6603	0.5541 0.5491 0.5592	2.3156 2.2976 2.3341
7	9.5627 9.9861 9.1325	14.1819 14.0504 14.3168	7.2012 6.9829 7.4371	0.5562 0.5510 0.5614	2.3809 2.3620 2.4003

Table 3. *Cont.*

λ^{-}	S_{av}	V_{av}	L_{av}	RR	LR
1.8	10.7410 11.1563 10.2401	13.5640 13.4797 13.6497	1.9193 1.9016 1.9375	0.5319 0.5286 0.5353	2.1369 2.1262 2.1479
2	10.7595 11.2146 10.2972	13.5374 13.4555 13.6204	1.7608 1.7460 1.7759	0.5309 0.5277 0.5341	2.2221 2.2115 2.2330
2.2	10.8145 11.2729 10.3507	13.5126 13.4331 13.5932	1.6206 1.6082 1.6332	0.5299 0.5268 0.5331	2.3018 2.2913 2.3136
2.4	10.8660 11.3259 10.4009	13.4897 13.4124 13.5680	1.4963 1.4859 1.5069	0.5290 0.5260 0.5321	2.3763 2.3659 2.3869
2.6	10.9142 11.3755 10.4478	13.4684 13.3931 13.5446	1.3859 1.3771 1.3949	0.5282 0.5252 0.5312	2.4459 2.4356 2.4563
2.8	10.9594 11.4220 10.4917	13.4486 13.3752 13.5229	1.2877 1.2802 1.2954	0.5274 0.5245 0.5303	2.5108 2.5007 2.5210
3	11.0016 11.4655 10.5328	13.4303 13.3586 13.5028	1.2002 1.1937 1.2068	0.5267 0.5239 0.5295	2.5713 2.5614 2.5814

Table 4. Performance measures vs. ν under RRP; $\lambda^+ = 5,\ \lambda^- = 1,\ \mu = 15,\ \kappa = 1.$

ν	S_{av}	V_{av}	L_{av}	RR	LR
1	10.4293 10.8777 9.9749	13.6926 13.5965 13.7905	2.7998 2.7612 2.8397	0.5370 0.5332 0.5408	1.7367 1.7260 1.7475
1.2	11.5715 12.0587 11.0774	12.4789 12.3874 12.5721	2.1985 2.1724 2.2255	0.5872 0.5829 0.5916	1.5968 1.5869 1.6070
1.4	12.5297 13.0489 12.0029	11.4631 11.3761 11.5518	1.8188 1.7996 1.8385	0.6293 0.6246 0.6342	1.4813 1.4720 1.4908
1.6	13.3451 13.8910 12.7907	10.6004 10.5177 10.6847	1.5601 1.5452 1.5753	0.6651 0.6599 0.6704	1.3844 1.3757 1.3933
1.8	14.0472 14.6140 13.4694	9.8585 9.7797 9.9387	1.3742 1.3623 1.3865	0.6959 0.6903 0.7016	1.3020 1.2937 1.3103
2	14.6581 15.2465 14.0602	9.2136 9.1385 9.2900	1.2352 1.2254 1.2454	0.7226 0.7167 0.7286	1.2310 1.2232 1.2389
2.2	15.1945 15.8000 14.5790	8.6478 8.5762 8.7208	1.1281 1.1198 1.1367	0.7461 0.7399 0.7524	1.1693 1.1619 1.1767
2.4	15.6692 16.2898 15.0383	8.1476 8.0791 8.2173	1.0435 1.0363 1.0508	0.7668 0.7604 0.7734	1.1151 1.1082 1.1222

Table 4. *Cont.*

ν	S_{av}	V_{av}	L_{av}	RR	LR
2.6	16.0924	7.7020	0.9752	0.7853	1.0673
	16.7261	7.6365	0.9690	0.7786	1.0607
	15.4478	7.7687	0.9816	0.7921	1.0740
2.8	16.4718	7.3026	0.9193	0.8018	1.0247
	17.1174	7.2398	0.9137	0.7950	1.0184
	15.8151	7.3665	0.9249	0.8089	1.0311
3	16.8141	6.9425	0.8727	0.8168	0.9865
	17.4703	6.8822	0.8678	0.8097	0.9805
	16.1464	7.0039	0.8777	0.8240	0.9926

Table 5. Performance measures vs. κ under RRP; $\lambda^{+} = 5,\ \lambda^{-} = 1,\ \mu = 15,\ \nu = 1$.

κ	S_{av}	V_{av}	L_{av}	RR	LR
1	10.4293	13.6926	2.7998	0.5370	1.7367
	10.8777	13.5965	2.7612	0.5332	1.7260
	9.9749	13.7905	2.8397	0.5408	1.7475
1.2	9.6443	14.7246	3.1521	0.5774	1.8486
	10.0636	14.6395	3.1099	0.5741	1.8390
	9.2205	14.8111	3.1958	0.5808	1.8583
1.4	8.9548	15.5927	3.5168	0.6115	1.9440
	9.3468	15.5167	3.4702	0.6085	1.9353
	8.5592	15.6699	3.5650	0.6145	1.9528
1.6	8.3492	16.3323	3.9015	0.6405	2.0264
	8.7164	16.2637	3.8496	0.6378	2.0184
	7.9791	16.4019	3.9553	0.6432	2.0345
1.8	7.8156	16.9696	4.3143	0.6655	2.0985
	8.1604	16.9072	4.2560	0.6630	2.0911
	7.4684	17.0327	4.3748	0.6680	2.1060
2	7.3432	17.5242	4.7640	0.6872	2.1622
	7.6679	17.4672	4.6979	0.6850	2.1554
	7.0166	17.5820	4.8327	0.6875	2.1692
2.2	6.9229	18.0112	5.2610	0.7063	2.2191
	7.2294	17.9587	5.1852	0.7043	2.2127
	6.6147	18.0644	5.3398	0.7084	2.2256
2.4	6.5469	18.4422	5.8178	0.7232	2.2704
	6.8371	18.3936	5.7302	0.7213	2.2643
	6.2553	18.4913	5.9090	0.7251	2.2765
2.6	6.2089	18.8262	6.4504	0.7383	2.3168
	6.4842	18.7810	6.3481	0.7365	2.3111
	5.9323	18.8719	6.5573	0.7401	2.3227
2.8	5.9035	19.1705	7.1797	0.7518	2.3593
	6.1655	19.1284	7.0586	0.7501	2.3538
	5.6405	19.2132	7.3065	0.7535	2.3648
3	5.6264	19.4810	8.0339	0.7640	2.3983
	5.8762	19.4415	7.8884	0.7624	2.3931
	5.3758	19.5210	8.1865	0.7655	2.4036

Now, we present the effect of initial parameters as well as considered schemas of changing probabilities α_m, $1 \leq m \leq S$ on the performance measures of the investigated RRP as follows:

- An analysis of data in Tables 2–5 showed that the second schema was favorable for all performance measures, except for the average inventory level. For the average inventory level, the third schema was favorable. It is interesting to note that the first schema was always intermediate between the three schemes.
- Table 2 shows that for all schemas, except for the average inventory level, performance measures increased versus the rate of consumer customers. These findings were expected.

- From Table 3, we can see that the average inventory level as well as the rate of loss of consumer customers increased when the rate of negative customers increased. However, the main performance measures decreased as the rate of negative customers increased. These findings were true for all schemas, and they were also expected.
- From Table 4, we can notice that the average inventory level as well as the reorder rate increased when the replenishment rate increased. A first observation concerning the behavior of reorder rate was unexpected. This phenomenon was explained as follows: when the replenishment rate increased, the probability that the inventory level was positive also increased and, hence, the catastrophe rate increased (see the second term in Formula (28)). Here, the rest of the performance measures were decreased versus replenishment rate. These findings were true for all schemas, and they were also expected.
- Table 5 shows that for all schemas, excluding the average inventory level, performance measures increased versus the rate of catastrophes. These findings were true for all schemas, and they were also expected.

Note that the values of all performance measures in all Tables 2–5 changed smoothly.

Under (s, Q) Policy

- Catastrophic events occur only in its warehouse part and they form the Poisson flow with the parameter κ. Upon arrival of catastrophic event, all the inventory is instantly destroyed, and even the item, which is at the status of release to the c-customer, is destroyed. The c-customer whose service was interrupted due to a catastrophe is returned to the queue, i.e. the catastrophe only destroys the inventory and does not force c-customers out of the system. If the inventory level is zero, then the catastrophe does not affect the operation of the system warehouse.
- Here (s, Q), $Q = S - s$, inventory replenishment policy is considered, i.e. when the inventory level drops to the reorder point s, where $0 \leq s < (S/2)$, an order of size $Q = S - s$ is placed for replenishment.
- The lead time of the order is exponentially distributed with mean ν^{-1}.

Ergodicity Condition

Let X_t be the number of c-customers at time t and Y_t be the inventory level at time t. Then, the process $Z_t = \{(X_t, Y_t), t \geq 0\}$ forms a continuous time two-dimensional Markov chain (2D MC) with state space

$$E = \bigcup_{n=0}^{\infty} L(n) \ ,$$

where $L(n) = \{(n, 0), (n, 1), ..., (n, S)\}$ is the subset of state space E with $X(t) = n$ called the level n.

Let $q((n_1, m_1), (n_2, m_2))$ denote the transition rate from state $(n_1, m_1) \in E$ to state $(n_2, m_2) \in E$. Taking into account the assumptions made in Sect. 2, we obtain following formulas for the generator $G = (q((n_1, m_1), (n_2, m_2))), (n_1, m_1), (n_2, m_2) \in E$:

$$q((n_1, m_1), (n_1 + 1,0)) = \lambda^+ \varphi_1 \cdot \chi(m_1 = 0); \qquad (32)$$

$$q((n_1, m_1), (n_1 + 1, m_1)) = \lambda^+ \cdot \chi(m_1 > 0); \tag{33}$$

$$q((n_1, m_1), (n_1 - 1, m_1)) = \lambda^- \cdot \chi(n_1 > 0); \tag{34}$$

$$q((n_1, m_1), (n_1 - 1, m_1 - 1)) = \mu \cdot \chi(n_1 > 0) \cdot \chi(m_1 > 0); \tag{35}$$

$$q((n_1, m_1), (n_1, 0)) = \kappa \cdot \chi(m_1 > 0); \tag{36}$$

$$q((n_1, m_1), (n_1, m_1 + S - s)) = \nu \cdot \chi(m_1 \leq s). \tag{37}$$

In (1)-(6) $\chi(A)$ is the indicator function of the event A, which is 1 if A is true and 0 otherwise. From relations (32)-(36) we conclude that the process $Z_t, t \geq 0$, is a Level Independent Quasi Birth-Death (LIQBD) process and its generator G might be represented as follows:

$$G = \begin{pmatrix} B & A_0 & O & \dots & O & \dots \\ A_2 & A_1 & A_0 & \cdots & O & \dots \\ O & A_2 & A_1 & A_0 & O & \dots \\ O & O & A_2 & A_1 & A_0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}, \tag{38}$$

where O denotes zero square matrix with dimension $S + 1$, and all other block matrices are square matrices of the same dimension.

Entities of the block matrices $B = \|b_{ij}\|$ and $A_k = \|a_{ij}^{(k)}\|, i, j = 0,1, ..., S,$ are determined from following relations:

$$b_{ij} = \begin{cases} \nu & \text{if } 0 \le i \le \text{s}, j = i + S - s, \\ \kappa & \text{if } i > 0, j = 0, \\ -(\nu + \lambda^+ \varphi_1) & \text{if } i = j = 0, \\ -(\nu + \kappa + \lambda^+) & \text{if } 0 < i \le s, i = j, \\ -(\kappa + \lambda^+) & \text{if } s < i \le S, i = j, \\ 0 & \text{in other cases;} \end{cases} \tag{39}$$

$$a_{ij}^{(0)} = \begin{cases} \lambda^+ \varphi_1 & \text{if } i = j = 0, \\ \lambda^+ & \text{if } i \ne 0, i = j, \\ 0 & \text{in other cases;} \end{cases} \tag{40}$$

$$a_{ij}^{(1)} = \begin{cases} \nu & \text{if } 0 \le i \le s, j = i + S - s, \\ \kappa & \text{if } i > 0, j = 0, \\ -(\lambda^- + \nu + \lambda^+ \varphi_1) & \text{if } i = j = 0, \\ -(\nu + \kappa + \mu + \lambda^+ + \lambda^-) & \text{if } 0 < i \le s, i = j, \\ -(\kappa + \mu + \lambda^+ + \lambda^-) & \text{if } i > s, i = j, \\ 0 & \text{in other cases;} \end{cases} \tag{41}$$

$$a_{ij}^{(2)} = \begin{cases} \lambda^- & \text{if } i = j, \\ \mu & \text{if } i > 0, j = i - 1, \\ 0 & \text{in other cases.} \end{cases} \tag{42}$$

The entities of the generator $A = A_0 + A_1 + A_2$ are determined as follows:

$$a_{ij} = \begin{cases} -\nu & \text{if } i = j = 0, \\ \nu & \text{if } 0 \le i \le s, j = i + S - s, \\ \mu + \kappa & \text{if } i = 1, j = 0, \\ \kappa & \text{if } i > 1, j = 0, \\ -(\mu + \nu + \kappa) & \text{if } 0 \le i \le s, j = i, \\ -(\mu + \kappa) & \text{if } i > s, j = i \\ \mu & \text{if } i \ge 2, j = i - 1. \end{cases} \tag{43}$$

We denote the steady-state probabilities that correspond to the finite generator matrix A by the vector $\boldsymbol{\pi} = (\pi(0), \pi(1), \ldots, \pi(S))$. The vector satisfies the following balance equations:

$$\boldsymbol{\pi} A = 0, \boldsymbol{\pi} \boldsymbol{e} = 1 \tag{44}$$

where $\mathbf{0}$ is null row vector of dimension $S + 1$ and $\boldsymbol{e}$ is column vector of dimension $S + 1$ that contains only 1's.

The balance equations in (44) can be rewritten as (see Fig. 2)

$$-\nu\pi(0) + (\kappa + \mu)\pi(1) + \kappa(\pi(2) + \cdots + \pi(S)) = 0 \tag{45}$$

$$-(\nu + \kappa + \mu)\pi(j) + \mu\pi(j+1) = 0, 1 \le j \le s \tag{46}$$

$$-(\kappa + \mu)\pi(j) + \mu\pi(j+1) = 0, s+1 \le j \le Q-1 \tag{47}$$

$$\nu\pi(j-Q) - (\kappa + \mu)\pi(j) + \mu\pi(j+1) = 0, Q \le j \le S-1 \tag{48}$$

$$\nu\pi(s) - (\kappa + \mu)\pi(S) = 0, \tag{49}$$

with the normalizing condition

$$\sum_{j=0}^{S} \pi(j) = 1. \tag{50}$$

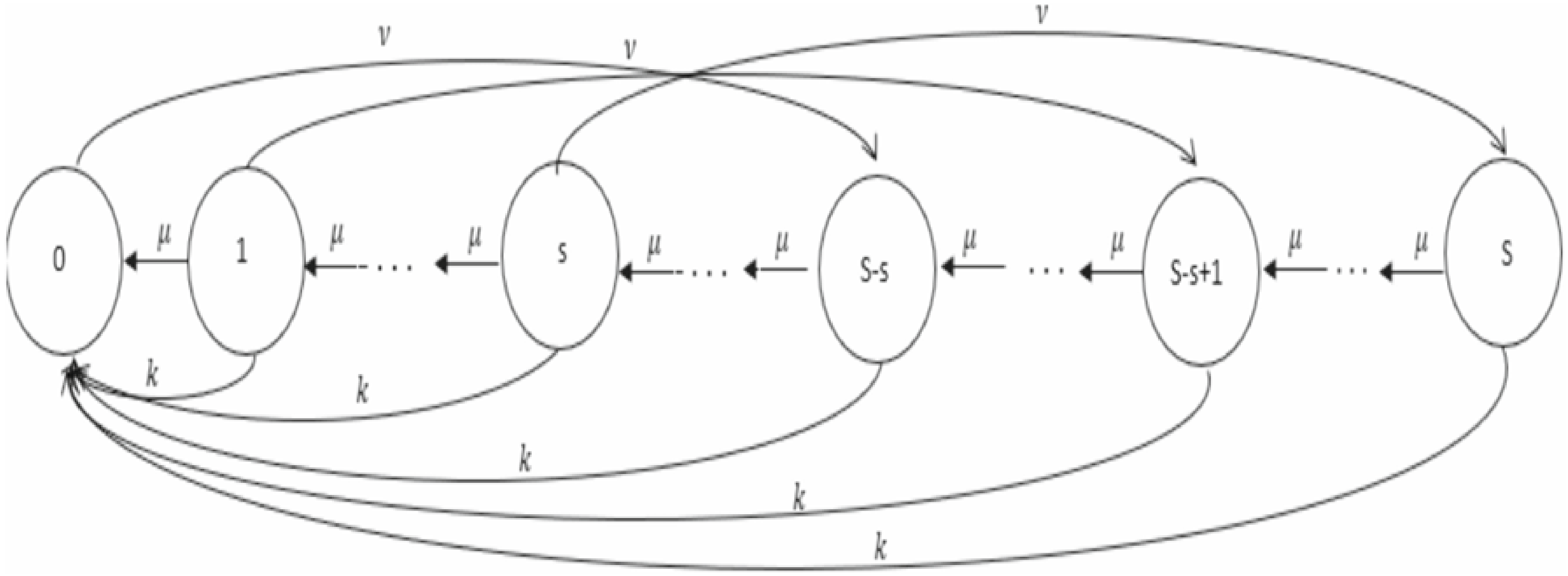


Fig. 2. State diagram

$$\pi(j+1) = \pi(1)a_{j+1}a_{j+1} = \left(1+\frac{\kappa}{\mu}\right)^{j-s}, s+1 \le j \le Q-1 \tag{51}$$

From the Eq. (48)

$$\pi(j+1) = \pi(1)a_{j+1} - \pi(0)b_{j+1}, Q \le j \le S-1 \tag{52}$$

where

$$a_{j+1} = a_{s+1}\left(1+\frac{\kappa}{\mu}\right)^{j-s} - \sum_{k=1}^{j-Q} a_k\left(1+\frac{\kappa}{\mu}\right)^{j-Q-k}\left(\frac{\nu}{\mu}\right) \text{ and } b_{j+1} = \left(\frac{\nu}{\mu}\right)\left(1+\frac{\kappa}{\mu}\right)^{j-Q}.$$

Then, by using the Eq. (45) , we write $\pi(1)$ in terms of $\pi(0)$ as following.

$$\pi(1) = \pi(0)\left(\frac{\kappa+\nu}{\mu}\right) - \left(\frac{\kappa}{\mu}\right). \tag{53}$$

We get the probability given in (54) by using the normalizing condition in (49) and he results

$$\pi(0) + \pi(1) + \pi(1)[a_2 + \cdots + a_{s+1}] + \pi(1)[a_{s+2} + \cdots + a_S] - \pi(0)[b_{Q+1} + \cdots + b_S] = 1$$

$$\pi(0)[1 - (b_{Q+1} + \cdots + b_S)] + \pi(1)[1 + (a_2 + \cdots + a_S)] = 1$$

$$\pi(0) = \frac{1 + \frac{\kappa}{\mu}\sum_{j=1}^{S} a_j}{1 + \left(\frac{\kappa+\nu}{\mu}\right)\sum_{j=1}^{S} a_j - \sum_{j=Q+1}^{S} b_j}. \tag{54}$$

That is,

$$\lambda^{+}\varphi_1\pi(0) + \lambda^{+}\sum_{j=1}^{S}\pi(j) < \lambda^{-} + \mu\sum_{j=1}^{S}\pi(j). \tag{55}$$

By using the normalizing condition (49) , we derive the following result:

$$\lambda^{+}(1 - \varphi_2\pi(0)) < \lambda^{-} + \mu(1 - \pi(0)). \tag{56}$$

Proposition. The process Z_t, $t \geq 0$, is ergodic if and only if the following condition is fulfilled:

$$\rho = \frac{\lambda^{+}(1 - \varphi_2\pi(0))}{\lambda^{-} + \mu(1 - \pi(0))} < 1. \tag{57}$$

Let $\boldsymbol{p} = (p_0, p_1, p_2, \dots)$ denote the steady-state probabilities of the queueing-inventory system that corresponds to the generator matrix G in (7), where $(S+1)$ dimensional vector p_n is partitioned as $p_n = (p(n,0), p(n,1), \dots, p(n,S)),\ n \geq 0$. That is, the vector $\boldsymbol{p}$ satisfies

$$\boldsymbol{p}G = 0 \text{ and } \boldsymbol{p}\boldsymbol{e} = 1. \tag{58}$$

Each part $p(n,i)$ gives the steady-state probability that there are $n, n \geq 0$, c-customers in the system and the number of items in the inventory is $i, 0 \leq i \leq S$.

Under the ergodicity condition (57), the steady-state probabilities of the queueing-inventory system are determined from the following equations:

$$p_n = p_0 R^n,\ n \geq 1, \tag{59}$$

where R is nonnegative minimal solution of the following quadratic matrix equation:

$$R^2 A_2 + R A_1 + A_0 = 0.$$

Bound probabilities p_0 are determined from following system of equations with normalizing condition:

$$p_0(B + RA_2) = 0.$$

$$p_0(I - R)^{-1} e = 1\ , \tag{60}$$

where I indicate the identity matrix of dimension $S+1$.

Performance Measures

The main performance measures of the investigated system related to both inventory and queueing are determined via steady-state probabilities. By using the standard technique, we can compute the mentioned performance measures of the investigated models as follows.

Performance measures related to inventory are following:

- Average inventory level (S_{av})

$$S_{av} = \sum_{m=1}^{S} m \sum_{n=0}^{\infty} p(n, m) \ ;$$

- Average order size

$$V_{av} = (S - s) \sum_{m=0}^{s} \sum_{n=0}^{\infty} p(m, n) \ ;$$

- Average reorder rate (RR)

$$RR = \mu \sum_{n=1}^{\infty} p(n, s+1) + \left(1 - \sum_{n=0}^{\infty} p(n, 0)\right) \ ;$$

Performance Measures

Performance measures related to queueing are following:

- The probability that there is no c-customer in the system (P_{idle})

$$P_{idle} = \sum_{m=0}^{S} p(0, m);$$

- Average length of queue (L_{av})

$$L_{av} = \sum_{n=1}^{\infty} n \sum_{m=0}^{S} p(n, m);$$

- Average loss rate of c-customers due to lack of stock (LR_1)

$$LR_1 = {}^{+}\varphi_2 \sum_{n=0}^{\infty} p(n, 0);$$

- Average loss rate of c-customers due to arriving of n-customers (LR_2)

$$LR_2 = {}^{-}\left(1 - \sum_{m=0}^{S} p(0, m)\right).$$

Numerical Results

Firstly, we discuss the behavior of the performance measures versus initial parameters under various scenarios in Table 1.

Towards this end, the reorder point and the maximum inventory level of the system are fixed by $s = 3$ and $S = 10$, respectively. The other parameters are vary as following; as the arrival rate λ^+ is varied, the others are fixed by $(\lambda^-, \mu, \nu, \kappa, \varphi_1) = (1,8, 1,1, 0.6)$; as the arrival rate λ^- is varied, the others are fixed by $(\lambda^+, \mu, \nu, \kappa, \varphi_1) = (5,8, 1,1, 0.6)$; as the service rate μ is varied, the others are fixed by $(\lambda^+, \lambda^-, \nu, \kappa, \varphi_1) = (5,1, 1,1, 0.6)$; as the replenishment rate ν is varied, the others are fixed by $(\lambda^+, \lambda^-, \mu, \kappa, \varphi_1) = (5,1, 8,1, 0.6)$; as the rate of the catastrophic events κ is varied, the others are fixed by $(\lambda^+, \lambda^-, \mu, \nu, \varphi_1) = (5,1, 8,1, 0.6)$; and the probability φ_1 is varied, the others are fixed by $(\lambda^+, \lambda^-, \mu, \nu, \kappa) = (4,1, 8,1, 1)$.

Numerical Results

Secondly, we provide an optimization discussion about inventory policy for some specific parameters. For this purpose, the function of the expected total cost, ETC, is structured as follows:

$$ETC = (c_k + c_r V_{av})RR + c_h S_{av} + c_{ps}\kappa S_{av} + c_l(LR_1 + LR_2) + c_w L_{av} \tag{61}$$

where

c_k: the fixed cost of one order;

c_r: the unit cost of the order size;

c_h: the holding cost per item in the inventory per unit of time,

c_l: the cost incurred due to the loss of a c-customer,

c_w: the waiting cost of a c-customer in the system,

c_{ps}: the damaging cost per item in the inventory.

Towards finding the optimum values of the reorder points (s^*) that minimize ETC, we vary the maximum inventory level by $S = 50, 70, 90$ and the values of the parameters

Table 1. Performance measures versus initial parameters.

Parameters		ρ	P_{idle}	L_{av}	LR_1	LR_2	S_{av}	RR	V_{av}
$\boldsymbol{\lambda^+}$	3.2	0.587	0.36	2.0211	0.6825	0.6390	2.4768	0.7112	4.3554
	3.6	0.661	0.29	2.7655	0.7844	0.7055	2.3729	0.7362	4.4831
	4	0.734	0.22	3.9212	0.8919	0.7709	2.2691	0.7586	4.6062
	4.4	0.808	0.16	5.9606	1.0054	0.8354	2.1657	0.7787	4.7249
	4.8	0.881	0.10	10.5296	1.1249	0.8991	2.0631	0.7968	4.8392
$\boldsymbol{\lambda^-}$	1	0.918	0.06	15.8998	1.1869	0.9306	2.0123	0.8051	4.8948
	1.8	0.768	0.20	4.2652	1.1281	1.4269	2.2184	0.7684	4.6635
	2.6	0.661	0.31	2.3216	1.0921	1.7797	2.3634	0.7373	4.4916
	3.4	0.580	0.40	1.5541	1.0697	2.0355	2.4682	0.7117	4.3629
	4.2	0.516	0.46	1.1561	1.0551	2.2267	2.5468	0.6910	4.2649
$\boldsymbol{\mu}$	7.6	0.945	0.04	23.9255	1.1849	0.9534	2.0213	0.8027	4.8830
	8.4	0.894	0.09	12.1043	1.1887	0.9098	2.0041	0.8072	4.9056
	9.2	0.851	0.12	8.4447	1.1919	0.8730	1.9896	0.8111	4.9247
	10	0.814	0.15	6.6645	1.1946	0.8415	1.9772	0.8144	4.9411
	10.8	0.783	0.18	5.6123	1.1969	0.8143	1.9665	0.8174	4.9553
$\boldsymbol{\nu}$	1	0.918	0.06	15.8998	1.1869	0.9306	2.0123	0.8051	4.8948
	1.8	0.756	0.22	3.7858	0.8117	0.7771	3.1789	1.0586	3.6612
	2.6	0.690	0.29	2.5228	0.6116	0.7076	3.8868	1.1911	2.9106
	3.4	0.655	0.33	2.0677	0.4892	0.6692	4.3578	1.2712	2.4116
	4.2	0.633	0.35	1.8408	0.4073	0.6454	4.6929	1.3245	2.0572
$\boldsymbol{\kappa}$	0.2	0.789	0.21	4.5078	0.7848	0.7826	2.9571	0.6114	4.0241
	0.4	0.822	0.17	5.9412	0.9182	0.8273	2.6401	0.6785	4.3105
	0.6	0.854	0.13	7.8949	1.0253	0.8659	2.3882	0.7305	4.5417
	0.8	0.886	0.10	10.8316	1.1133	0.9000	2.1830	0.7717	4.7331
	1	0.918	0.06	15.8998	1.1869	0.9306	2.0123	0.8051	4.8948
$\boldsymbol{\varphi_1}$	0.1	0.437	0.56	0.7730	1.8993	0.4342	2.5716	0.6833	4.2382
	0.3	0.556	0.40	1.4358	1.4993	0.5937	2.4623	0.7129	4.3692
	0.5	0.675	0.28	2.7452	1.0976	0.7165	2.3379	0.7437	4.5237
	0.7	0.794	0.17	5.8538	0.6808	0.8230	2.1955	0.7731	4.6917
	0.9	0.913	0.07	18.2507	0.2363	0.9246	2.0334	0.8014	4.8724

in Table 2. For this study, we fix the unit values of the defined above costs by $c_k = 10$, $c_r = 15$, $c_h = 10$, $c_{ps} = 15$, $c_l = 450$ and $c_w = 400$.

Table 2. The optimum values of the reorder point and the expected total cost

Values of parameters						$S = 50$		$S = 70$		$S = 90$	
λ^+	λ^-	μ	ν	κ	φ_1	ETC^*	s^*	ETC^*	s^*	ETC^*	s^*
6	1	16	1	1	0.6	2694.7911	17	2870.8960	26	3045.4455	35
7	1	16	1	1	0.6	3425.0770	16	3596.1270	25	3771.9921	34
8	1	16	1	1	0.6	4530.6860	15	4676.5780	24	4850.2452	33
9	1	16	1	1	0.6	6558.4322	12	6599.8848	23	6753.7845	33
6	**2**	8	1	1	0.6	4227.9575	18	4403.4035	28	4576.3051	37
6	**3**	8	1	1	0.6	3185.0078	20	3359.7394	28	3531.7546	38
6	**4**	8	1	1	0.6	2902.9759	20	3075.7552	29	3246.7151	38
6	**5**	8	1	1	0.6	2804.2382	20	2977.0169	29	3147.1726	39
6	1	**8**	1	1	0.6	18706.8655	16	18605.2363	27	18755.321	36
6	1	**10**	1	1	0.6	4728.6918	17	4896.7306	26	0 5071.5662	36
6	1	**12**	1	1	0.6	3448.8654	17	3623.1019	26	3798.0826	35
6	1	**14**	1	1	0.6	2957.7823	17	3133.4054	26	3308.1553	35
6	1	8	**1**	1	0.6	18706.8655	16	18605.2363	27	18755.321	36
6	1	8	**2**	1	0.6	3778.7009	18	3996.8951	27	0 4211.1519	36
6	1	8	**3**	1	0.6	2871.8851	18	3106.4792	27	3336.9669	36
6	1	8	**4**	1	0.6	2544.7377	18	2788.0840	27	3027.3981	36
5	1	16	1	**1**	0.6	2152.8606	18	2328.1694	27	2500.9895	36
5	1	16	1	**2**	0.6	3087.5624	20	3298.8199	29	3508.4478	39
5	1	16	1	**3**	0.6	4118.8440	20	4351.1046	30	4583.1957	39
5	1	16	1	**4**	0.6	5841.0924	21	6089.0218	30	6333.3897	40
6	1	16	1	1	**0.3**	2250.6602	18	2427.1020	27	2601.0389	36
6	1	16	1	1	**0.5**	2495.4774	17	2672.0209	26	2845.9296	36
6	1	16	1	1	**0.7**	2963.8109	17	3138.0746	26	3312.8409	35
6	1	16	1	1	**0.9**	3792.8360	16	3951.0096	25	4125.0646	34

Under Base Stock Policy

The main assumptions of the QIS model studied here are as follows:

- The maximum warehouse capacities equal to S, S $<\infty$.
- Homogeneous and positive c-customers arrive at the facility with one server according to the Poisson process at a rate of λ^+, and each c-customer needs a stock of unit size. The waiting room for c-customers has an infinite size.
- Consumer customers from the queue are selected for servicing according to their arrivals and their service times are assumed to be exponential with parameter μ.
- Along with c-customers, the system receives n-customers with rate λ^-. The influence of n-customers is as follows: (1) If at the moment of arrival of the n-customer, there is a queue of c-customers, then one c-customer is pushed out from the queue; (2) A n-customer can force out from the system even a c-customer, which is in the server if a queue is empty. In such cases the inventory level does not change, i.e. it is assumed that stocks are released after the completion of servicing a c-customer; (3) If there are no c-customers in the system, then the received n-customer does not affect the operation of the system.

- The hybrid sales scheme is used, i.e. if there are no stocks in the system upon arrival of c-customer, then, in accordance to the Bernoulli trials, it either with probability (w.p.) φ_1 joins the queue of infinite length(backorder sale scheme), or w.p. φ_2 leaves the system unserved(lost sale scheme), where $\varphi_1 + \varphi_2 = 1$. If the stock level is positive, then the arriving c-customer is queued w.p. 1.
- Catastrophes follow the Poisson flow with the rate κ, and at the moment of arrival of such an event, all the items in the stock are instantly destroyed. As a result of the catastrophes, even the item, which is at the status of release to the c-customer, is destroyed, and the c-customer whose service was interrupted is returned to the queue; in other words, the catastrophe only destroys the stocks of the system and does not force c-customers out of the system. If the inventory level is zero, then the disaster does not affect the operation of the system.
- Base replenishment policy is used. It means that a replenishment is called every time an item sells out and lead times are assumed to be exponential with parameter ν, $\nu < \infty$.

Stationary Distribution

Let X_t be the number of c-customers at the time and Y_t be the inventory level at the time. Then, the process $\{Z_t, t \geq 0\} = \{(X_t, Y_t), t \geq 0\}$ forms a continuous time Markov chain (CTMC) with state space E $= \{0, 1, \dots\} \times \{0, 1, \dots, S\}$.

Proposition 1. The generator G of the process $\{Z_t, t \geq 0\}$ has following form:

$$G = \begin{pmatrix} B & A_0 & O & \dots & O & \dots \\ A_2 & A_1 & A_0 & O & O & \dots \\ O & A_2 & A_1 & A_0 & O & \dots \\ O & O & A_2 & A_1 & A_0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix} \qquad (62)$$

where O denotes zero square matrices with dimension S+1, and all other block matrices are square matrices of the same dimension. Entities of the block matrices

$B = \| b_{ij} \|$ and $A_k = \| a_{ij}^{(k)} \|$, i,j = 0,1, . . . , S are given by

$$b_{ij} = \begin{cases} (S-i)\nu \text{ if } \quad 0 \le i \le S-1, \quad j = i+1 \\ \kappa \text{ if } \quad 0 < i \le S, \quad j = 0 \\ -(S\nu + \lambda^+ \varphi_1) \text{ if } \quad i = j = 0 \\ -\Big((S-i)\nu + k + \lambda^+\Big) \text{ if } \quad 0 < i \le S, \quad i = j, \\ 0 \quad \text{in other cases;} \end{cases} \tag{63}$$

$$a_{ij}^{(0)} = \begin{cases} \lambda^+ \varphi_1 \text{ if } \quad i = j = 0 \\ \lambda^+ \text{ if } \quad i \ne 0, \quad i = j \\ 0 \quad \text{in other cases;} \end{cases} \tag{64}$$

$$a_{ij}^{(1)} = \begin{cases} (S-i)\nu \text{ if } \quad 0 \le i \le S-1, \quad j = i+1 \\ \kappa \text{ if } \quad i > 0, \quad j = 0 \\ -(\lambda^- + S\nu + \lambda^+ \varphi_1) \text{ if } \quad i = j = 0 \\ -\Big((S-i)\nu \ + \kappa + \mu + \lambda^+ + \lambda^-\Big) \text{ if } \quad 0 < i \le S, \quad i = j, \\ 0 \quad \text{in other cases;} \end{cases} \tag{65}$$

$$a_{ij}^{(2)} = \begin{cases} \lambda^- \text{ if } \quad i = j \\ \mu \text{ if } \quad > 0, \quad j = i-1 \\ 0 \quad \text{in other cases.} \end{cases} \tag{66}$$

Proof. The transition rate $(n_1, m_1) \to (n_2, m_2\ )$is denoted as $q((n_1, m_1);(n_2, m_2))$. By taking into account the assumptions made in Sect. 2, we conclude that the indicated parameters are calculated as follows:

(a) Transitions due to the arrival of c-customers:

$$\begin{aligned}&(n_1, m_1) \to (n_1 + 1, 0) : \text{the rate is} \quad \lambda^+\varphi_1, \quad \text{for} \quad m_1 = 0;\\&(n_1, m_1) \to (n_1 + 1, m_1) : \text{the rate is} \quad \lambda^+, \text{for} \quad 0 < m_1 \leq S.\end{aligned}$$

(b) Transitions due to the arrival of n-customers:

$(n_1, m_1) \to (n_1 - 1, m_1)$: the rate is λ^-, for $n_1 > 0$.

(c) Transitions due to service completion of c-customers $(n_1, m_1) \to (n_1 - 1, m_1 - 1)$: the rate is μ, for $n_1 > 0, m_1 > 0$.

(d) Transitions due to catastrophes: $(n_1, m_1) \to (n_1, 0)$: the rate is κ, for $m_1 > 0$.

(e) Transitions due to replenishment: $(n_1, m_1) \to (n_1, m_1 + 1)$: the rate is $(S - m_1)\nu$, for $0 \leq m_1 < S$

All other transition pairs have a rate of zero.

So, we have the following relations:

$$q((n_1,m_1);(n_1+1,0)) = \lambda^+\varphi_1 \cdot \chi(m_1=0); \tag{67}$$

$$q((n_1,m_1);(n_1+1,m_1)) = \lambda^+ \cdot \chi(m_1>0); \tag{68}$$

$$q((n_1,m_1);(n_1-1,m_1)) = \lambda^- \cdot \chi(n_1>0); \tag{69}$$

$$q((n_1,m_1);(n_1-1,m_1-1)) = \mu \cdot \chi(n_1>0) \cdot \chi(m_1>0); \tag{70}$$

$$q((n_1,m_1);(n_1,0)) = \kappa \cdot \chi(m_1>0); \tag{71}$$

$$q((n_1,m_1);(n_1,m_1+1)) = (S-m_1)\nu \cdot \chi(0 \le m_1<S). \tag{72}$$

Hereinafter, χ (A) is the indicator function of the event A, which is 1 if A is true and 0 otherwise. By considering a lexicographic order of the system's states, (0,0),(0,1),. . .,(0,S);(1,0),(1,1),. . .,(1,S); . . . ; $(i,0),(i,1),\ldots(i,S);\ldots$from relations (67)-(72) we conclude that the generator G of the process $Z_t, t \ge 0$ might be represent via relations (62)-(66)

Proposition 2. The process $\{Z_t, t \geq 0\}$ is ergodic if and only if the following condition is fulfilled:

$$\lambda^{+}(1-\varphi_2\pi(0))<\lambda^{-}+\mu(1-\pi(0)), \tag{73}$$

where

$$\pi(m)=b_m\left(\sum_{i=0}^{S} b_i\right)^{-1} \tag{74}$$

and parameters b_m, m = 0, 1, ..., S, are calculated via the following reverse recursive formulas:

$$b_m=\begin{cases}\frac{1}{(S-m)\nu}(a_{S-m-1}b_{m+1}-\mu b_{m+2}) \text{ if } \quad 0\leq m<S-2\\ \frac{1}{2\nu}(a_{S-m-1}b_{m+1}-\mu) \text{ if } \quad m=S-2\\ \frac{a_0}{\nu} \quad \text{ if } \quad m=S-1,\\ 1 \text{ if } \quad m=S;\end{cases} \tag{75}$$

$a_n=\mu+\kappa+n\nu, n=1,2,\ldots,S-1;$

Proof. By Neuts (1981), pp. 81-83, the process $\{Z_t, t \geq 0\}$ is ergodic if and only if

$$\pi A_0 \mathbf{e} < \pi A_2 \mathbf{e}, \tag{76}$$

where $\pi = (\pi(0), \pi(1), \ldots, \pi(S))$, is the stationary probability vector that correspond to generator $A = A_0 + A_1 + A_2$ and e is column vector of dimension S+1 that contains only 1 s.

From relations (64)-(66) conclude that the nonzero entities of the matrix A are determined as follows:

$$a_{ij} = \begin{cases} -S\nu \text{ if } & i = j = 0, \\ (S-i)\nu \quad \text{if} & 0 < i \leq S-1, \quad j = i+1, \\ \mu + \kappa \text{ if } & i = 1,\ j = 0, \\ \kappa \text{ if } & i > 1, \quad j = 0, \\ -\mu \text{ if } & i > 0, \quad j = 1, \\ \mu \text{ if } & i \geq 2, \quad j = i-1. \end{cases} \tag{77}$$

In other words, we have balance equations for stationary probability vector π :

$$\pi A = \mathbf{0}, \pi e = 1, \tag{78}$$

where $\mathbf{0}$ is the null row vector of dimension S+1.

Note 2. Consider the following special cases.

(i) If $\varphi_2 = 1$ (i.e. when purely lost sale scheme is used) and $\lambda^- = 0$ (i.e. when there are not n-customers) from (73) we find the ergodicity condition for the single-server Markovian queuing system, i.e., $\lambda^+ < \mu$, i.e. in this case, the ergodicity condition of the system does not depend on the storage size of the system, the rate of catastrophes, and the replenishment rate.

(ii) If $\varphi_2 = 1$ and $\lambda^- > 0$ then the ergodicity condition is depending on all indicated parameters of the system, see formulas (75)

(iii) If $\varphi_2 = 0$ (pure backorder scheme is used) then the ergodicity condition is depending on all indicated parameters of the system even for case $\lambda^- = 0$, see formulas (75)

Steady-state probabilities that correspond to the generator matrix G we denote by $p = (p_0, p_1, p_2, \ldots)$, where $p_n = (p(n,0), p(n,1), ..., p(n,S)), n = 0, 1, \ldots$. Under the ergodicity condition (73) desired steady-state probabilities are determined from the following equations:

$$p_n = p_0 R^n, n \geq 1, \tag{79}$$

where R is the non-negative minimal solution of the following quadratic matrix equation:

$$R^2 A_2 + R A_1 + A_0 = 0.$$

Performance Measures

Main performance measures can be divided into two groups: stock-related metrics and queuing-related metrics. Stock-related metrics are the following:

- Average inventory level ($S_{a\nu}$)

$$S_{a\nu} = \sum_{m=1}^{S} m \sum_{n=0}^{\infty} p(n,m) \tag{19}$$

- Destruction rate of the stocks (DRS):

$$DRS = \kappa(1 - \sum_{n=0}^{\infty} p(n,0)) \tag{20}$$

- Average reorder rate (RR)

$$RR = \kappa \sum_{m=1}^{S} p(0,m) + (\mu + \kappa) \sum_{n=1}^{\infty} \sum_{m=1}^{S} p(n,m); \tag{21}$$

Performance Measures

Main performance measures can be divided into two groups: stock-related metrics and queuing-related metrics. Stock-related metrics are the following:

- Average inventory level ($S_{a\nu}$)

$$S_{a\nu} = \sum_{m=1}^{S} m \sum_{n=0}^{\infty} p(n, m) \tag{80}$$

- Destruction rate of the stocks (DRS):

$$DRS = \kappa(1 - \sum_{n=0}^{\infty} p(n, 0)) \tag{81}$$

- Average reorder rate (RR)

$$RR = \kappa \sum_{m=1}^{S} p(0, m) + (\mu + \kappa) \sum_{n=1}^{\infty} \sum_{m=1}^{S} p(n, m); \tag{82}$$

Performance Measures

Queuing-related metrics are the following:

- Loss rate (LR) of c-customers

$$LR = \lambda^{+}\varphi_2 \sum_{n=0}^{\infty} p(n,0) + \lambda^{-}\left(1 - \sum_{m=0}^{S} p(0,m)\right) \tag{83}$$

- Average length of the queue of c-customers ($L_{a\nu}$)

$$L_{a\nu} = \sum_{n=1}^{\infty} n \sum_{m=0}^{S} p(n,m) \tag{84}$$

Numerical Results

In Tables 1 through 6, we display the behavior of main system performance measures as well as Expected Total Cost (ETC) versus initial parameters. ETC is defined as follows:

$$ETC = K \cdot RR + c_h S_{av} + c_d DRS \cdot S_{a\nu} + c_l LR + c_w L_{a\nu}, \tag{24}$$

where K is the fixed price of one order, c_h is the price of unit inventory holding per unit of time, c_d is the price of unit inventory destruction, c_l is the cost for a single c-customer loss, c_w is the price per unit time of queuing delay for a single c-customer.

In all our examples we take S = 50 and values of other parameters are shown in the table's titles. The coefficients in the expression for the functional in ETC (see (25)) were chosen as follows: K = 10, $c_h = 20$, $c_l = 10$, $c_w = 20$, $c_d = 40$.

Due to the limited volume of the paper, a detailed analysis of the results of numerical experiments is left to the reader. Here we briefly analyze the presented tables.

Common to all tables is the conclusion that all performance measures as well as ETC change smoothly. We register the following observations from these tables.

Numerical Results

In Tables 1 through 6, we display the behavior of main system performance measures as well as Expected Total Cost (ETC) versus initial parameters. ETC is defined as follows:

$$ETC = K \cdot RR + c_h S_{av} + c_d DRS \cdot S_{a\nu} + c_l LR + c_w L_{a\nu}, \tag{85}$$

where K is the fixed price of one order, c_h is the price of unit inventory holding per unit of time, c_d is the price of unit inventory destruction, c_l is the cost for a single c-customer loss, c_w is the price per unit time of queuing delay for a single c-customer.

In all our examples we take S = 50 and values of other parameters are shown in the table's titles. The coefficients in the expression for the functional in ETC (see (86)) were chosen as follows: K = 10, c_h = 20, c_l = 10, c_w = 20, c_d = 40.

Due to the limited volume of the paper, a detailed analysis of the results of numerical experiments is left to the reader. Here we briefly analyze the presented tables.

Common to all tables is the conclusion that all performance measures as well as ETC change smoothly. We register the following observations from these tables.

Numerical Results

- An increase λ^+ results in a decrease(with very slow rate) in measures $S_{a\nu}$, DRS and ETC; other measures are increasing versus λ^+ (see Table 1).
- Measures $S_{a\nu}$, DRS and ETC are increasing(with very slow rate)ones versus λ^-; other measures are decrease (see Table 2).
- An increase μ results in a decrease in measures $S_{a\nu}$, DRS (with very slow rate), $L_{a\nu}$, ETC; other measures areal mos constants (see Table 3).
- $S_{a\nu}$, ETC increase strongly compared to ν, and DRS and RR also increase, but very slowly; other measures are decreasing (see Table 4).
- An increase in κ leads to strong changes in all stock-related indicators and ETC, and only $S_{a\nu}$ decreases compared to κ; queue-related metrics are increasing at a moderate rate (see Table 5).
- All performance measures are almost constants versus φ_1, only RR and $L_{a\nu}$ are increased at a very slow rate (see Table 6).

Table 1. Effect of λ^+ on the performance measures; $\lambda^- = 2$, $\kappa = 3$, $\mu = 10$, $\nu = 3$, $\varphi_1 = 0.6$.

λ^+	$S_{a\nu}$	DRS	RR	LR	$L_{a\nu}$	ETC
5	24.3136	2.9395	5.9222	0.0202	0.7256	3418.97
5.2	24.2861	2.9394	6.1198	0.0202	0.7772	3418.14
5.4	24.2587	2.9393	6.3174	0.0202	0.8319	3417.37
5.6	24.2312	2.9393	6.5151	0.0202	0.8901	3416.66
5.8	24.2037	2.9392	6.7126	0.203	0.9522	3416.04
6	24.1763	2.9391	6.9102	0.0203	1.0185	3415.50
6.2	24.1489	2.9391	7.1078	0.0203	1.0894	3415.05
6.4	24.1214	2.9390	7.3054	0.0203	1.1655	3414.71
6.6	24.0941	2.9389	7.503	0.0204	1.2474	3414.48
6.8	24.0665	2.9389	7.7006	0.0204	1.3357	3414.38
7	24.0391	2.9388	7.8981	0.0204	1.4312	3414.42

Table 2. Effect of λ^- on the performance measures; $\lambda^+ = 6$, $\kappa = 3$, $\mu = 10$, $\nu = 3$, $\varphi_1 = 0.6$.

λ^-	$S_{a\nu}$	DRS	RR	LR	$L_{a\nu}$	ETC
1	24.1000	2.9389	7.4594	0.0204	1.2289	3414.52
1.2	24.1164	2.9390	7.3416	0.0203	1.1801	3414.66
1.4	24.1322	2.9390	7.2279	0.0203	1.1351	3414.83
1.6	24.1474	2.9391	7.1183	0.0203	1.0934	3415.03
1.8	24.1621	2.9391	7.0125	0.0203	1.0546	3415.26
2	24.1763	2.9391	6.9102	0.0203	1.0185	3415.51
2.2	24.1901	2.9392	6.8114	0.0203	0.9848	3415.76
2.4	24.2033	2.9392	6.7158	0.0203	0.9532	3416.03
2.6	24.2162	2.9392	6.6233	0.0203	0.9236	3416.32
2.8	24.2286	2.9393	6.5337	0.0202	0.8958	3416.61
3	24.2407	2.9393	6.4469	0.0202	0.8696	3416.91

Table 3. Effect of μ on the performance measures; $\lambda^+ = 6$, $\kappa = 3$, $\lambda^- = 2$, $\nu = 3$, $\varphi_1 = 0.6$.

μ	$S_{a\nu}$	DRS	RR	LR	$L_{a\nu}$	ETC
9	24.1883	2.9392	6.9088	0.0203	1.1761	3420.32
9.2	24.1851	2.9392	6.9092	0.0203	1.1323	3419.00
9.4	24.1822	2.9392	6.9096	0.0203	1.0916	3417.77
9.6	24.1791	2.9391	6.9099	0.0203	1.0538	3416.60
9.8	24.1763	2.9392	6.9102	0.203	1.0185	3415.50
10	24.1736	2.9391	6.9105	0.0203	0.9855	3414.46
10.2	24.1709	2.9391	6.9108	0.0203	0.9545	3413.47
10.4	24.1683	2.9391	6.9111	0.0203	0.9255	3412.53
10.6	24.1658	2.9391	6.9114	0.0203	0.8981	3411.63
10.8	24.1634	2.9391	6.9117	0.0203	0.8724	3410.78
11	24.1611	2.9391	6.9121	0.0203	0.8481	3409.97

Table 4. Effect of ν on the performance measures; $\lambda^+ = 6$, $\kappa = 3$, $\lambda^- = 2$, $\mu = 10$, $\varphi_1 = 0.6$.

ν	$S_{a\nu}$	DRS	RR	LR	$L_{a\nu}$	ETC
1	11.2953	2.8121	6.7197	0.0626	1.0631	1585.54
1.2	13.1308	2.8445	6.7687	0.0518	1.0509	1845.88
1.4	14.8016	2.8674	6.8032	0.0442	1.0427	2083.06
1.6	16.3284	2.8844	6.8287	0.0385	1.0367	2299.9
1.8	17.6057	2.8901	6.8456	0.0342	1.0301	2423.5
2	19.0176	2.9081	6.864	0.0306	1.0286	2682.03
2.2	20.2076	2.9166	6.8767	0.0278	1.0258	2851.19
2.4	21.3098	2.9337	6.8872	0.0254	1.0235	3007.88
2.6	22.3334	2.9296	6.8961	0.0235	1.0215	3153.43
2.8	23.2866	2.9347	6.9037	0.0218	1.0199	3288.97
3	24.1763	2.9391	6.9102	0.0203	1.0185	3415.50

Table 5. Effect of k on the performance measures; $\lambda^{+} = 6$, $\lambda^{-} = 2$, $\nu = 3$, $\mu = 10$, $\varphi_1 = 0.6$.

κ	$S_{a\nu}$	DRS	RR	LR	$L_{a\nu}$	ETC
1	34.4452	1.2001	6.0726	0.0083	1.0074	2423.34
1.2	32.9607	1.3866	6.162	0.0096	1.0086	2569.23
1.4	31.4601	1.5923	6.2608	0.0112	1.0099	2715.87
1.6	30.2156	1.7779	6.3499	0.0123	1.0111	2836.97
1.8	28.9499	1.9825	6.4483	0.0137	1.0124	2959.58
2	27.8928	2.1671	6.5372	0.0157	1.0135	3061.49
2.2	26.8604	2.3609	6.6307	0.0163	1.0148	3160.55
2.4	25.9018	2.5542	6.724	0.0176	1.0161	3252.07
2.6	25.0093	2.7469	6.8172	0.0191	1.0173	3336.82
2.8	24.1763	2.9391	6.9102	0.0203	1.0185	3414.38
3	23.397	3.1309	7.0031	0.0216	1.0197	3488.68

Table 6. Effect of φ_1 on the performance measures; $\lambda^+ = 6$, $\lambda^- = 2$, $\kappa = 3$, $\nu = 3$, $\mu = 10$.

φ_1	$S_{a\nu}$	DRS	RR	LR	$L_{a\nu}$	ETC
0	24.1863	2.9392	6.8381	0.0203	0.9931	3415.72
0.1	24.1847	2.9392	6.8501	0.0203	0.9972	3415.68
0.2	24.1831	2.9392	6.8621	0.0203	1.0014	3415.64
0.3	24.1813	2.9392	6.8741	0.0203	1.0056	3415.60
0.4	24.1796	2.9392	6.8862	0.0203	1.0099	3415.57
0.5	24.178	2.9392	6.8982	0.0203	1.0142	3415.53
0.6	24.1763	2.9391	6.9102	0.0203	1.0185	3415.50
0.7	24.1746	2.9391	6.9223	0.0203	1.0229	3415.46
0.8	24.1732	2.9391	6.9343	0.0203	1.0273	3415.43
0.9	24.1713	2.939	6.9464	0.0203	1.0317	3415.40
1	24.1696	2.9391	6.9584	0.0203	1.0362	3415.37

Models of Finite Queuing-Inventory Systems Under (s, S) Policy

Consider a single-server finite QIS in which the warehouse has a maximum capacity S. Arriving homogeneous c-customers are represented by a Poisson flow with intensity λ^+. Customer homogeneity means that each customer requires the same amount of inventory. The service times of the c-customers are independent identically distributed (i.i.d.) random variables with an exponential cumulative distribution function (c.d.f.); its mean value is equal to μ^{-1} and the inventory level decreases by one unit when c-customer service ends. The waiting room for queuing c-customers has a finite size $R, R < \infty$. This means that if, when a c-customer arrives, the buffer is completely occupied, then the arriving new c-customer is lost with probability (w.p.) 1; otherwise, the arriving c-customer will enter the buffer if the server is busy. A combined sales scheme is applied, i.e., if upon the arrival of a c-customer, the warehouse is empty, then, in accordance with the Bernoulli trials, the customer either enters the buffer w.p. φ_1 or leaves the system without items w.p. $\varphi_2 = 1 - \varphi_1$.

In addition to c-customers, the system also receives negative customers (n-customers) with intensity λ^-. Negative customers require no service or inventory, but upon the arrival of such customers, one c-customer is pushed out of the system, if any. The detailed procedure of managing the pushing out of the c-customer is as follows: (1) if there is a queue of c-customers, then only the c-customer is pushed out of the queue; (2) if there is no queue of c-customers and only the c-customer is receiving service, then the n-customer evicts the c-customer, which is located in the server, from the system (in these cases the inventory level remains the same since items are released after the completion of servicing a c-customer); (3) if there are no c-customers in the system (in buffer or on the server), then the arrived n-customer does not impact the operation of the system.

Catastrophes are represented by a Poisson flow with intensity κ, and when a catastrophe occurs, all inventory is instantly destroyed. The catastrophe destroys even the items that are allocated for sale to the c-customer. In this case, the interrupted c-customer returns to the buffer, i.e., the catastrophe only destroys the items and does not push out the c-customer from the system. Catastrophes do not affect the operation of the warehouse if it is empty.

In order to be specific, here, $(s,\ S)$ is the inventory replenishment policy considered (sometimes this policy is called "Up to S" as well). This means that when the inventory level drops to the re-order point s, $0 \leq s < S$, a replenishment order is placed, and upon replenishment, the inventory level is restored to level S, regardless of how many items were in inventory.

The lead times of the replenishment's i.i.d. variables with exponential c.d.f. are represented by the average value of the lead times, which is equal to ν^{-1}.

The problem is to find the joint distribution of the number of c-customers in the system and the inventory level in the warehouse, as well as to calculate the main performance measures: the mean number of items in the warehouse, the mean order size, and the mean re-order rate, which includes the mean length of the queue and the loss rate of c-customers.

Steady-State Analysis An Exact Approach

This subsection proposes an exact method for obtaining the steady-state probabilities and the main performance measures defined above. As in Melikov et al. (2023) [12], let X_t be the number of c-customers at time t and Y_t be the inventory level at time t. So, the process $Z_t = \{(X_t, Y_t), t \geq 0\}$ forms a two-dimensional continuous-time Markov chain (2D CTMC) with the following state space:

$$E = \bigcup_{m=0}^{S} E_m \tag{86}$$

where $E_m = \{(0, m), (1, m), \cdots, (R, m)\}$ is the subset of states in which the inventory level is equal to $m, m = 0, 1, \cdots, S$.

The transition rate from micro-state (n_1, m_1) to micro-state (n_2, m_2) is denoted by $q((n_1, m_1), (n_2, m_2))$. By taking into account the assumptions related to operating the investigated QIS, we obtain the following relations to determine these transition rates:

$$q((n_1,m_1),(n_2,m_2)) = \begin{cases} \lambda^+\varphi_1, & m_2 = m_1 = 0, n_2 = n_1 + 1, \\ \lambda^+, & m_2 = m_1 > 0, n_2 = n_1 + 1, \\ \lambda^-, & m_2 = m_1, n_2 = n_1 - 1, \\ \mu, & m_2 = m_1 - 1, n_2 = n_1 - 1, \\ \kappa, & m_1 > 0, m_2 = 0, n_2 = n_1, \\ \nu, & m_1 \le s, m_2 = S, n_2 = n_1. \end{cases} \quad (87)$$

From relations (2) we conclude that each state of the constructed 2D CTMC can be reached from any other state through a finite number of transitions, i.e., the considered chain is an irreducible one. In other words, for each positive value of the loading parameters, a steady-state regime exists. Let us denote by (n, m) the probability of the state $(n, m) \in E$. The desired steady-state probabilities are obtained as a solution of the system of balance equations (SBE), constructed using relations (87)

Case 1: When $(n, 0) \in E_0$, the following is true:

$$\begin{gathered}(\lambda^+\varphi_1\chi(n < R) + \lambda^-\chi(n > 0) + \nu)\,p(n,0) = \lambda^+\varphi_1 p(n-1,0)\chi(n > 0) \\ +\lambda^- p(n+1,0)\chi(n < R) + \mu p(n+1,1)\chi(n < R) + \kappa\textstyle\sum_{m=1}^{S} p(n,m).\end{gathered} \quad (88)$$

Case 2: When $(n, m) \in E_m, 0 < m \leq s$, the following is true:

$$\begin{gathered}(\lambda^{+}\chi(n < R) + \lambda^{-}\chi(n > 0) + \nu + \mu + \kappa)p(n, m) = \lambda^{+}p(n-1, m)\chi(n > 0) \\ +\lambda^{-}p(n+1, m)\chi(n < R) + \mu p(n+1, m+1)\chi(n < R).\end{gathered} \tag{89}$$

Case 3: When $(n, m) \in E_m, s < m < S$, the following is true:

$$\begin{gathered}(\lambda^{+}\chi(n < R) + \lambda^{-}\chi(n > 0) + \mu + \kappa)p(n, m) = \lambda^{+}p(n-1, m)\chi(n > 0) \\ +\lambda^{-}p(n+1, m)\chi(n < R) + \mu p(n+1, m+1)\chi(n < R).\end{gathered} \tag{90}$$

Case 4: When $(n, S) \in E_S$, the following is true:

$$\begin{gathered}(\lambda^{+}\chi(n < R) + \lambda^{-}\chi(n > 0) + \mu + \kappa)p(n, S) = \lambda^{+}p(n-1, S)\chi(n > 0) \\ +\lambda^{-}p(n+1, S)\chi(n < R) + \mu p(n+1, m+1)\chi(n < R) + \nu\sum_{m=0}^{s} p(n, m).\end{gathered} \tag{91}$$

Here and below, $\chi(A)$ is the indicator function of the event A, i.e., it is equal to 1 if A is true; otherwise, it is equal to 0. A normalization condition should be added to SBE (88)-(91), i.e., the following is true:

$$\sum_{(n,m)\in E} p(n, m) = 1. \tag{92}$$

The constructed SBE(88)-(92)is a system of linear algebraic equations of dimension $(R+1)\cdot(S+1)$, and it can be solved numerically using known software if the QIS has moderate buffer and storage sizes.

After determining the steady-state probabilities, the main characteristics of the QIS under study can be calculated using a standard technique. These characteristics are divided into two groups: (1) inventory-related performance measures and (2) queuing-related performance measures. The first group of characteristics includes the mean number of items in the warehouse (S_{av}), the mean order size (V_{av}), and the mean re-order rate (RR).

- The mean number of items in the warehouse (i.e., the average inventory level) is calculated as a mathematical expectation of the appropriate random variable and is given by the following:

$$S_{av} = \sum_{m=1}^{S} m \sum_{n=0}^{R} p(n, m) . \tag{93}$$

- Similar to (93) the average order size (i.e., the average size of replenished items from external source) is calculated as a mathematical expectation of the appropriate random variable and is calculated as follows:

$$V_{av} = \sum_{m=S-s}^{S} m \sum_{n=0}^{R} p(n, S-m)\,. \tag{94}$$

- An inventory order is placed in two cases: (1) if the inventory level drops to the re-order point s after completing customer service in states $(n, s+1) \in E_{s+1}$, or (2) if catastrophes occur in the states $(n, m) \in E_m,\ m > 0$. Therefore, the average reorder intensity is calculated as follows:

$$RR = \mu \sum_{n=1}^{R} p(n, s+1) + \kappa\left(1 - \sum_{n=0}^{R} p(n, 0)\right). \tag{95}$$

The second group of performance measures includes the average length of the queue (L_{av}) and loss rate of c-customers (LR).

- The mean length of the queue is calculated as a mathematical expectation (an average value) of the appropriate random variable and is given by the following:

$$L_{av} = \sum_{n=1}^{R} n \sum_{m=0}^{S} p(n,m)\ . \tag{96}$$

- Losing c-customers occurs in three cases: (1) if, at the time the c-customer arrives, the waiting room is full (with probability 1), i.e., the system is in one of the states $(R,m) \in E_m$, $m = 0,1,\cdots,S$; (2) if, at the time the c-customer arrives, the inventory level is zero and the waiting room is not full (with probability φ_2), i.e., the system is in one of the states $(n,0) \in E_0$, $n < R$; (3) when an n-customer arrives, it displaces one c-customer. Therefore, the loss rate of c-customers is calculated as follows:

$$LR = \lambda^{+} \sum_{m=0}^{S} p(R,m) + \lambda^{+} \varphi_2 \sum_{n=0}^{R-1} p(n,0) + \lambda^{-} \left(1 - \sum_{m=0}^{S} p(0,m)\right). \tag{97}$$

An Approximate Approach

In this subsection, we derive the closed-form approximate solution for the steady-state probabilities of the investigated 2D CTMC by using a space merging approach
This approach is highly accurate for systems with rare catastrophes, i.e., it is assumed that $\kappa \ll min(\lambda^+, \lambda^-, \mu)$. Note that the last assumption is not extraordinary, since in the opposite case (i.e., when the rate of catastrophes is close to the rate of c-customers, the speed of their service, and the rate of n-customers), the QIS under consideration is generally not effective.

In the case where the above assumption is fulfilled, the basic requirement for an adequate application of the space-merging method is satisfied. In this case, transition rates between states in each subset E_m (see (1)) are much greater than the transition rates between states from different subsets. So, in accordance with the space merging algorithm, a subset of states E_m in (1) is combined into one merged state $< m >$, and the merging function in the initial state space (1) is defined as follows: $U(n, m) =< m >$, $(n, m) \in E$. The merged states constitute the set $\hat{E} = \{< m >: m = 0, 1, \ldots, S\}$. Then, to calculate the approximate values of steady-state probabilities, $\hat{p}(m, n)$, we have the following formula:

$$\hat{p}(n, m) \approx \rho_m(n)\pi(< m >) \tag{98}$$

where $\rho_m(n)$ denotes the probability of state (n,m) within subset E_m and $\pi(<m>)$ denotes the probability of merged state $<m>\in \hat{E}$.

From relations (2), we conclude that the state probabilities $\rho_0(n)$, $n=0,1,\cdots,R$ within a split model with the state space E_0 coincide with the distribution of a finite birth–death process in which the birth rate is $\lambda^+\varphi_1$, while the death rate is λ^-. In the same way, from relations (2), we conclude that the state probabilities $\rho_m(n)$, $m>0, n=0,1,\cdots,R$ within a split model with the state space E_m are independent of m and coincide with the distribution of a finite birth–death process in which the birth rate is λ^+, while the death rate is λ^-. In other words, state probabilities within split models are determined as follows:

$$\rho_m(n) = \begin{cases} \theta^n \frac{1-\theta}{1-\theta^{R+1}}, & m>0, n=0,1,\ldots,\mathrm{R} \\ \theta_0^n \frac{1-\theta_0}{1-\theta_0^{R+1}}, & m=0, n=0,1,\ldots,\mathrm{R} \end{cases} \tag{99}$$

where $\theta_0 = \lambda^+\varphi_1/\lambda^-$ and $\theta = \theta_0/\varphi_1$.

Note 1. *To simplify the notation, for cases $m > 0$ below, the subscript m is omitted in state probabilities $\rho_m(n)$. In cases where $\theta = 1$ and/or $\theta_0 = 1$, all state probabilities $\rho_m(n) = 1/(R+1)$ for each $n,\ n = 0, 1, \ldots, R$ and $m,\ m = 0, 1, \ldots, S$.*

Let us denote the transition rate from the merged state $< m_1 >$ to the merged state $< m_2 >$ by $q(< m_1 >, < m_2 >)$. Then, taking into account relations (2) and (14), we propose the following formulas for determining these rates (all other transition rates are zero):

Case $0 \le m \le s$:

$$q(< m >, < S >) = \nu \sum_{n=0}^{R} \rho_m(n) = \nu. \tag{100}$$

Case $m > 0$:

$$q(< m >, < 0 >) = \kappa \sum_{n=0}^{R} \rho_m(n) = \kappa; \tag{101}$$

$$q(< m >, < m-1 >) = \mu \sum_{n=1}^{R} \rho_m(n) = \mu(1 - \rho(0)). \tag{102}$$

In other words, the merged model represents a one-dimensional Markov chain in state space $\hat{E}$ where transition rates between merged states are calculated via Formulas (100)-(101) Using the approach proposed in (96) we develop the following closed-form formulas for calculating the probabilities of merged states:

$$\pi(0) = \frac{1+bc}{1+dc}, \tag{103}$$

$$\pi(1) = d\pi(0) - b, \tag{104}$$

$$\pi(m) = a_m\pi(1), 2 \le m \le S, \tag{105}$$

where the following statements are true:

$$d = \tfrac{\nu+\kappa}{\mu(1-\rho(0))},\ b = \tfrac{\kappa}{\mu(1-\rho(0))},$$
$$c = \textstyle\sum_{m=1}^{S} a_m,\ a_m = \begin{cases} (1+d)^{m-1}, & \text{if } 1 \le m \le s+1, \\ (1+d)^s(1+b)^{m-s-1}, & \text{if } s+1 < m \le S. \end{cases}$$

Eventually, taking into account Formulas (13), (14), and (18)–(20), we conclude that the approximate values of performance measures (8)–(12) can be calculated using the following explicit formulas:

$$S_{av} = \textstyle\sum_{m=1}^{S} m\pi(m); \tag{106}$$

$$V_{av} = \textstyle\sum_{m=S-s}^{S} m\pi(S-m); \tag{107}$$

$$RR = \mu(1-\rho(0))\pi(s+1) + \kappa(1-\pi(0)); \tag{108}$$

$$L_{av} = \textstyle\sum_{n=1}^{R} n(\rho_0(n)\pi(0) + \rho(n)(1-\pi(0))); \tag{109}$$

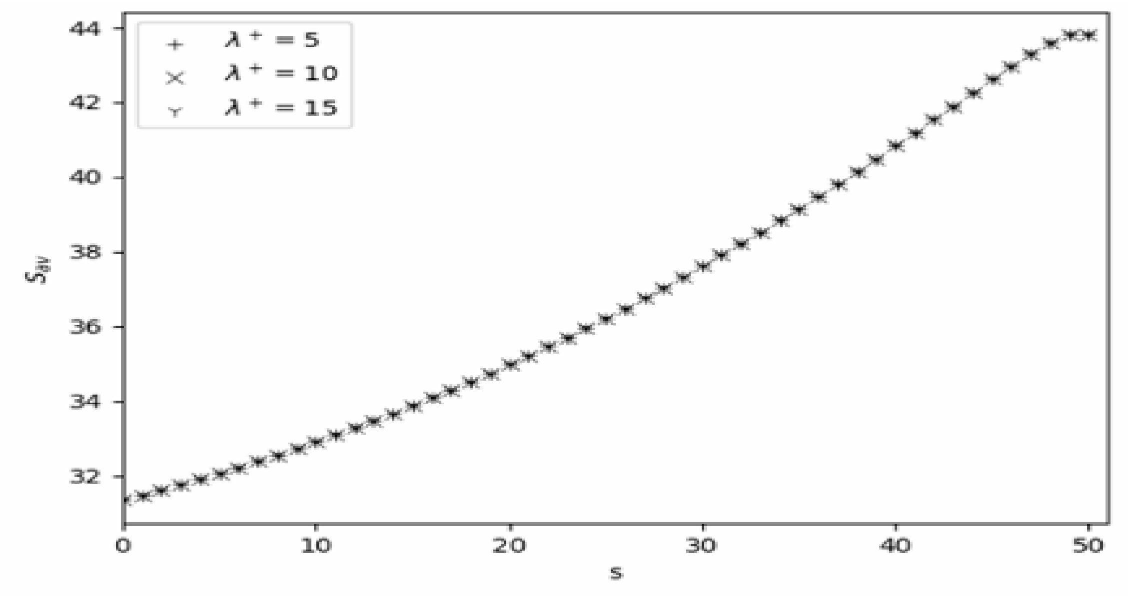
$$\begin{gathered} LR = \lambda^+\varphi_2\pi(0)(1-\rho_0(0)) + \lambda^+\rho_0(R)\pi(0) + \lambda^+(1-\pi(0))\rho(R) + \\ \lambda^-(\pi(0)(1-\rho_0(0)) + (1-\pi(0))(1-\rho(0))) \end{gathered} \tag{110}$$

Numerical Experiments

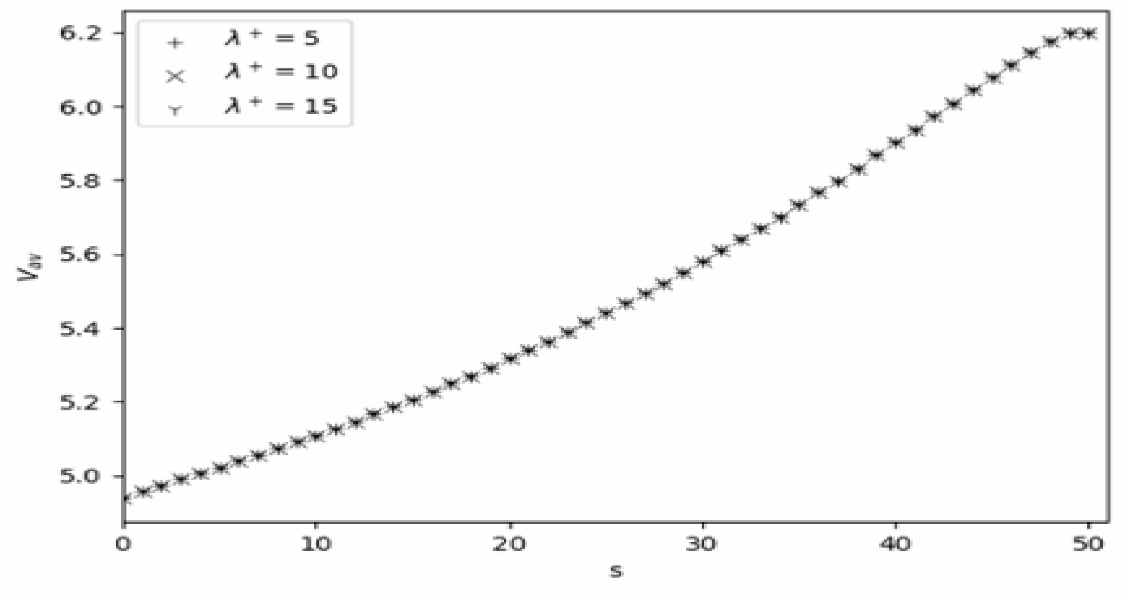
The accuracy of the proposed approximate formulas is investigated via numerical evaluations. For this purpose, exact values of the steady-state probabilities (SSP) are determined from SBE (3)–(7) for the QIS with a maximum capacity of warehouse $S = 50$ and buffer size $R = 30$, where the dimension of SBE is equal to 1581. The accuracy of the developed approximate formulas can be estimated using several norms, e.g., cosine similarity, Euclidean distance, Jaccard norm, etc. To be specific, here, we use a simple norm, that is, the maximum errors when calculating SSPs. Some results of numerical evaluations are shown in Table 1. In this table, along with an indication of the accuracy of calculating the SSPs, results are given that indicate the accuracy of calculating the performance measures (8)–(12). From this table, we conclude that the accuracy of the proposed approximate formulas for calculating SSPs and performance indicators is high for engineering applications. From this table, it is also clear that the accuracy of calculating the SSPs is greater than the accuracy of calculating performance indicators. This was to be expected, since the performance indicators are calculated through SSPs using operations of multiplication by large numbers; see Formulas (8)–(12) and (21)–(25). We conducted a large number of experiments and summarize only a small part of them here. An interesting result of these experiments is that the larger the system size (i.e., increasing S and R), the higher the accuracy of the approximate results obtained.

Table 1. Dependence of the absolute error of the SSPs and performance measures vs. s; $\lambda^+ = 15$, $\lambda^- = 1, \mu = 2, \kappa = 0.1, \nu = 1, \varphi_1 = 0.4$.

s	Max of Error for SSPs	Error for				
		S_{av}	V_{av}	RR	L_{av}	LR
0	1.17×10^{-3}	7.01×10^{-2}	1.12×10^{-1}	1.23×10^{-2}	1.41×10^{-1}	1.54×10^{-2}
5	1.02×10^{-3}	6.05×10^{-1}	1.13×10^{-2}	1.05×10^{-2}	1.02×10^{-1}	1.27×10^{-2}
10	2.15×10^{-3}	3.11×10^{-2}	2.29×10^{-2}	1.91×10^{-2}	1.17×10^{-1}	1.36×10^{-2}
15	8.77×10^{-4}	4.02×10^{-2}	3.01×10^{-2}	5.14×10^{-2}	1.43×10^{-1}	1.78×10^{-2}
20	7.01×10^{-4}	3.18×10^{-1}	6.08×10^{-2}	4.02×10^{-2}	2.01×10^{-1}	2.15×10^{-2}
25	3.73×10^{-3}	5.02×10^{-2}	7.11×10^{-2}	2.72×10^{-2}	2.02×10^{-1}	3.01×10^{-2}
30	2.16×10^{-3}	1.08×10^{-2}	4.33×10^{-2}	5.05×10^{-2}	1.04×10^{-1}	2.02×10^{-2}
35	2.41×10^{-3}	3.13×10^{-1}	1.02×10^{-1}	1.92×10^{-2}	1.51×10^{-1}	1.32×10^{-2}
40	1.24×10^{-3}	1.02×10^{-1}	8.12×10^{-2}	1.82×10^{-2}	1.11×10^{-1}	1.03×10^{-2}
45	3.45×10^{-3}	1.03×10^{-1}	2.01×10^{-1}	1.09×10^{-2}	1.21×10^{-1}	1.17×10^{-2}

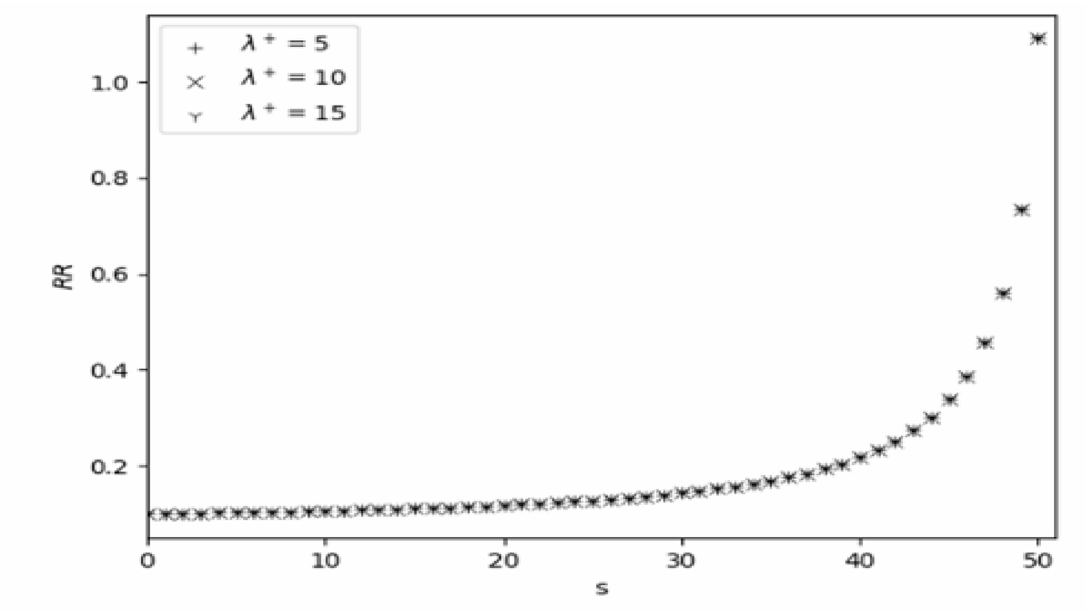


(a)



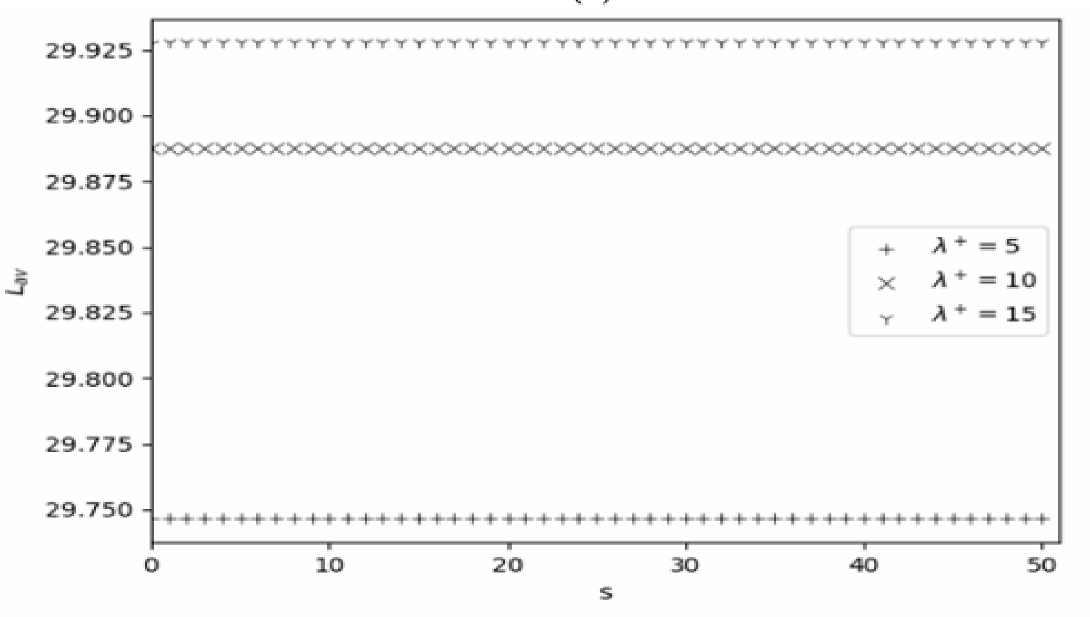
(b)

Figure 1. *Cont.*



λ + = 5
λ + = 10
λ + = 15
RR
1.0
0.8
0.6
0.4
0.2
0
10
20
30
40
50
s

(**c**)



29.925
29.900
29.875
29.850
29.825
29.800
29.775
29.750
λ + = 5
λ + = 10
λ + = 15
0
10
20
30
40
50
s

(**d**)

Optimization Problem

The third goal of performing numerical experiments is solving the optimization problem. To be specific, here, the minimization of Expected Total Cost (ETC) is considered. In this problem, it is assumed that all load parameters and structural parameters of the QIS are fixed, and the only controllable parameter is the reorder point. Similar to Melikov et al. (2023) [11], ETC is defined as follows:

$$ETC(s) = (K + c_r \cdot V_{av}) \cdot RR + c_h \cdot S_{av} + c_{ps} \cdot \kappa \cdot S_{av} + c_l \cdot LR + c_w \cdot L_{av} \quad (26)$$

where K is the fixed price of one order, c_r is the unit price of the order size, c_h is the unit item storage price per unit of time, c_{ps} is the price of unit item destruction, c_l is the cost for a single consumer customer loss, and c_w is the price per unit time of delay for a single consumer customer.

The problem is to find a value (optimal) of s that minimizes (26). For any values of initial parameters, this problem has a solution, since the admissible set for values of s is finite and discrete, i.e., $0 \leq s \leq S - 1$. Coefficients in (26) for the hypothetical model are selected as $K = 10, c_r = 15, c_h = 10, c_l = 450, c_w = 400$, and $c_{ps} = 15$. Some results of the minimization of (26) are demonstrated in Table 2. Here, we assume that $N = 30$, $\varphi_1 = 0.4$, $\lambda^+ = 15$, $\lambda^- = 1$, $\kappa = 0.1, \mu = 2$, and $\nu = 1$. The optimal solution for indicated values of S is $s^* = 0$. For completeness, Table 2 shows the values of the

Some directions for further research

I. QIS with state (both queue and inventory level)-dependent RPs.

Melikov A., Chakravarthy S.R. A new admission control scheme in queuing-inventory system with two priority classes of demands // OPSEARCH. 2024. doi.org/10.1007/s12597-024-00877-8.

Melikov A., Rumyantsev A. State-Dependent Admission Control in Heterogeneous Queueing-Inventory System with Constant Retrial Rate // Lecture Notes on Computer Science. 2025. V. 15460. P. 156-170.

Melikov A., Ozkar S. Algorithmic approach to study queueing-inventory systems with queue-dependent hybrid replenishment policy // Communications in Computer and Information Science. 2025. V. 2472.

Melikov A., Ozkar S. Analysis of queuing-inventory system with state-dependent replenishment policy // Operations Research. An International Journal. 2025. (In press)

Some directions for further research

II. QIS with multiple sources for replenishment.

Melikov A., Lawrence S. Sivakumar B. Analysis and optimization of hybrid replenishment policy in a double-sources queueing-inventory system with MAP arrivals // Annals of Operations Research. 2023. V. 331. Iss. 2. P. 1249-1267.

III. QIS with various types of customers (priorities, various size of inventory requirements, etc.,)

Otten S., Daduna H. Stability of queueing-inventory systems with customers of different priorities. Annals of Operations Research. 2023. V. 331. Iss. 2. P. 963–983.

- ***Thanks for attention***